DYNAMIC METEOROLOGY

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PREFACE

The great progress of meteorology in recent years has been largely due to the application of the laws of thermodynamics and hydrodynamics to the study of the atmosphere and its motions. It is the aim of this book to give an account of these investigations and their results, with regard to applications to weather forecasting and to research.

No previous knowledge of meteorology is assumed, although some preliminary training in general meteorology will facilitate the study of the book. A large number of references to literature have been given in order to enable the reader to consult the original papers. The material presented has been the subject of lecture courses on Dynamic Meteorology given at the University of Toronto during the past six years as part of the meteorological course offered by the university in cooperation with the Meteorological Service of Canada. The scope of the book is, in the main, a theoretical discussion of the various phenomena, without a complete descriptive account of the observed phenomena and of the actual practical applications of the theory. The mathematical technique has been kept as simple as possible. Readers who are sufficiently well versed in advanced mathematical methods will know how to obtain solutions for many of the specific problems discussed here by more elegant mathematical methods. Thus, the derivation of the equations of motion on the rotating earth (Sec. 45) could be shortened greatly by the use of vector analysis. Where more advanced results of thermodynamics or of hydrodynamics are used, they have been explained briefly, but the reader will do well to remember that this book does not deal with these subjects but with dynamic meteorology and that for a thorough study of thermodynamical or hydrodynamical problems, specialized textbooks should be consulted.

The problems are chosen partly to supplement the text with material of secondary importance and partly to indicate the possibilities of practical applications.

The formulas are numbered according to the decimal system. The number before the period refers to the section in which the formula appears, the number after the period indicates the position of the formula in the section. The formula with the smaller number comes first. Thus (17.21) precedes (17.3), but follows (17.2).

The author is indebted to Dr. W. Elsasser for permission to reproduce Fig. 21, to the editors of *Nature* for permission to reproduce Fig. 24, to Prof. J. Bjerknes for permission to reproduce Figs. 45, 79, 80, 86, 89, to Prof. S. Petterssen for permission to reproduce Figs. 55 to 57, to Sir Napier Shaw and Messrs. Constable and Co. for permission to reproduce Fig. 81, and to Mr. C. M. Penner and the National Research Council of Canada for permission to reproduce Figs. 84 and 85. Owing to the present war, it has been impossible to approach all the authors and publishers concerned for permission to reproduce diagrams which appeared in their publications. The author offers his apologies for this omission and hopes that the permission may be considered as granted, since proper references are made in each case and since all these diagrams have originally appeared in scientific journals.

The author wishes to express his gratitude to Prof. J. Patterson, controller of the Meteorological Service of Canada, to Mr. A. Thomson, assistant controller of the same service, to Prof. C. F. Brooks, director of Blue Hill Observatory, and to Prof. Sverre Petterssen, head of the Meteorological Department of the Massachusetts Institute of Technology, for their encouragement during the preparation of this book.

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CONTENTS

DYNAMIC METEOROLOGY

CHAPTER I

THE EARTH. THE EQUATION OF STATE FOR DRY AND MOIST AIR

1. The Earth and Its Gravitational Field. The earth is approximately a sphere or, more accurately, a spheroid with an equatorial radius of 6378.4 km and a polar radius of 6356.9 km. For almost all meteorological problems the deviation of the earth from the spherical form may be disregarded, so that the earth may be assumed as exactly spherical with a radius of 6371 km, approximately. A sphere of this radius has roughly the same area and volume as the earth.

The angular velocity of the earth's rotation

$$\omega = \frac{2\pi}{\text{sidereal day}} = 7.292 \times 10^{-5}\ \text{sec}^{-1}.$$

The acceleration of gravity that is observed on the earth consists in the actual attraction by the earth diminished by the effect of the centrifugal acceleration caused by the earth's rotation. Points near the equator move faster than those at higher latitudes owing to the earth's rotation. Therefore, the centrifugal force decreases poleward, and consequently the total acceleration of gravity increases. Moreover, owing to the spheroidal shape of the earth, points at higher latitudes are closer to the center of the earth. This is an additional reason for the increase of the acceleration of gravity poleward, for the gravitational force at a point outside the earth is inversely proportional to the distance from the center. The total acceleration can be expressed by the following formula for the acceleration of gravity at sea level g_0 and at latitude φ:

$$g_0 = 980.621(1 - 0.00264 \cos 2\varphi)\ \text{cm/sec}^2 \qquad (1.1)$$

Because the acceleration of gravity decreases with the square of the distance from the center, its value g at an altitude z above sea level is given by

$$g = \frac{g_0}{[1 + (z/E)]^2} \tag{1.2}$$

or $g \sim g_0(1 - 3.14 \times 10^{-7}z)$ if z is expressed in meters. $E = 6371$ km, the mean radius of the earth.

On mountains, Eq. (1.2) should be replaced by another equation, owing to the mass of the mountain and the imperfect isostatic compensation. The consideration of these corrections would lead too far into geodesy and is not of great importance to the meteorologist who finds these figures in tables.[1]

The height z of a point above sea level can also be expressed by the difference between the potential of gravity at sea level and at the altitude z. The potential at the altitude z is numerically equal to the work done when the unit of mass is lifted from sea level up to this height. It is called the geopotential. The following relation exists between the geopotential ψ and the height z:

$$\psi = \int_0^z g\,dz \tag{1.3}$$

according to which, with (1.2),

$$\psi = g_0 E^2 \int_0^z \frac{dz}{(z + E)^2} = g_0 \frac{z}{1 + (z/E)} \tag{1.31}$$

Because $z \ll E$, the denominator on the right side of this last equation is very nearly unity so that numerically ψ is about 10 times larger than z if the meter is used as the unit of length. In order to obtain approximate numerical equality between the geopotential and the corresponding altitude the former is usually expressed in a unit that is 10 times smaller than the one following from Eq. (1.31). This unit is called the "dynamic meter" or "geodynamic meter." It should be clearly understood that the dynamic meter is not an altitude but rather an energy per unit mass. The hundredth part of the dynamic meter is a dynamic centimeter, 1000 dynamic meters are a dynamic kilo-

[1] "Smithsonian Meteorological Tables," 5th ed., Smithsonian Institution, Washington, D. C., 1931. BJERKNES, V., "Dynamic Meteorology and Hydrography," Tables 1M and 2M, Carnegie Institution of Washington, Washington, 1910.

meter, etc. If the height above sea level is expressed in these units, it is called "dynamic height" to distinguish it from the ordinary geometric height. Obviously the following relation exists between the dynamic height ψ and the geometric height z:

$$\psi = \frac{g_0}{10} \frac{z}{1 + (z/E)} \tag{1.4}$$

Because $g_0 = 9.8$ m/sec² [if g_0 is expressed in centimeter-gram-second (cgs) units the factor $1/10$ has to be replaced by $1/1000$], ψ is about 2 per cent smaller numerically than z. With the aid of (1.4) and (1.1), dynamic heights and geometric heights may be transformed one into the other. In meteorological practice where speed is essential, tables are used for this transformation.[1]

The practical advantage of the dynamic height ψ over the geometric height z is due to the possibility of combining the variations of the acceleration of gravity g with the variable ψ which measures the elevation (see Sec. 6).

Dynamically, the surfaces of equal potential are more important than the surfaces of equal height because the force of gravity is everywhere normal to the former while it has a component parallel to the latter. Therefore, a sphere would be in equilibrium on a surface of equal potential but would roll toward the equator on a surface of constant height.

The surfaces of equal geometric and dynamic height intersect each other, but the inclination is small. The equipotential surface 20,000 dyn. meters, for instance, descends 107 m from the equator to the pole.

2. Units of Pressure, Temperature, and Density. Pressure is defined as the force exerted on the unit area. The unit of force in the cgs system being the dyne, it follows that the unit of pressure in the cgs system

$$\text{Dynes/cm}^2 = \text{gm cm}^{-1} \text{ sec}^{-2}$$

This quantity is too small for practical use in meteorology. A pressure of 10^6 cgs units has been called "1 bar."

$$1 \text{ bar} = 10^6 \text{ dynes/cm}^2$$

[1] BJERKNES, *op. cit.*, Tables 3M–6M. "Smithsonian Meteorological Tables," Tables 64–68. LINKE, F., "Meteorologisches Taschenbuch," I, Tables 26–27, Akademische Verlagsgesellschaft, Leipzig, 1931.

and in practice the millibar, *i.e.*, the thousandth part of a bar, is in most countries used as the unit in which the atmospheric pressure is expressed

$$1 \text{ mb} = 10^3 \text{ dynes/cm}^2$$

In addition to the millibar the following expressions are sometimes used:

$$1 \text{ decibar} = 10^{-1} \text{ bar}$$
$$1 \text{ centibar} = 10^{-2} \text{ bar}$$
$$1 \text{ microbar} = 10^{-6} \text{ bar}$$

It may be noted that the centibar is the unit of pressure in the meter-ton-second system.

In practice the atmospheric pressure is most frequently determined by the height of a mercury column exerting the same pressure as the air. Consequently, the pressure observations are given in units of length, millimeters or inches. Because the density of mercury is 13.6 and the acceleration of gravity at sea level and 45° latitude is 980.6 cm/sec², the pressure of a mercury column of height 1 mm in cgs units is

$$1 \text{ mm Hg} = 10^{-1} \times 13.6 \times 980.6 = 1333 \text{ dynes/cm}^2 = 1.333 \text{ mb}$$

Similarly, the pressure of a mercury column of height 1 in. is, because 1 in. = 25.4 mm,

$$1 \text{ in. Hg} = 33.86 \text{ mb}$$

The following scales are used to express temperature: According to the centigrade scale, the freezing and boiling points of water at "normal" atmospheric pressure (760 mm Hg = 1013 mb) have the values 0° and 100°, respectively. According to the Fahrenheit scale, these two fixed points have the values 32° and 212°. The relation between the two scales is therefore

$$t°\text{C} = \tfrac{5}{9}(t°\text{F} - 32) \tag{2.1}$$

The Réaumur scale according to which the freezing point of water is 0° and its boiling point 80° is today not used in meteorology. According to the absolute temperature scale the freezing point of water has the value 273°[1] and the boiling point 373°, so that the absolute temperature T is, in degrees centigrade,

$$T = t°\text{C} + 273° \tag{2.2}$$

[1] This figure is sufficiently accurate for all meteorological problems.

For a discussion of the theoretical foundations of the absolute temperature scale the reader is referred to the textbooks on thermodynamics.

The density ρ is defined as mass per unit volume. Its unit in the cgs system is gm/cm³. The specific volume v is the volume per unit mass. It is obviously

$$v = \frac{1}{\rho} \tag{2.3}$$

3. The Composition of the Atmosphere. Atmospheric air is a mixture of various gases. The two main constituents in the lower layers are nitrogen and oxygen which account for 99 per cent of volume and mass of the air. A critical survey by Paneth[1] shows the composition of the air near the surface to be as given in the following table in abbreviated form:

Gas	Volume, per cent	Molecular weight	Density, air = 1	Mass, per cent
Nitrogen	78.09	28.016	0.9670	75.51
Oxygen	20.95	32.000	1.1053	23.15
Argon	0 93	39.944	1.379	1.28
Carbon dioxide (variable)	0.03	44.000	1 529	0.046

There are also small traces of neon, helium, krypton, xenon, ozone, radon, and perhaps hydrogen present.

The table refers to completely dry air. The water vapor of the air is variable, for water may freeze, condense, and evaporate at the temperatures encountered in the atmosphere. It therefore requires separate consideration (Sec. 5).

The observations indicate that the composition of the atmosphere remains virtually unchanged at least up to 20 km. Ozone becomes more abundant at greater heights, with a maximum between 20 and 30 km. It has great influence upon the emission and absorption of radiation in the upper atmosphere, but its amount is not sufficient to affect the density of the air directly. At greater altitudes, but probably not below 100 km, lighter gases must become predominant.[2] For the problems of dynamic

[1] Paneth, F. A., *Quart. J. Roy. Met. Soc.*, **65**, 304, 1939.

[2] Chapman, S., and Milne, E. A., *Quart. J. Roy. Met. Soc.*, **46**, 357, 1928. Haurwitz, B., The Physical State of the Upper Atmosphere, *J. Roy. Astr. Soc. Can.*, 1937, 1938. Chapman, S., and Price, W. C., Report on Progress in Physics, *Phys. Soc. London*, **3**, 42, 1937.

meteorology the state of the high atmosphere is not important, at least according to our present knowledge.

4. The Gas Equation for Dry Atmospheric Air. In thermodynamics, it is shown that the following relation exists between pressure p, density ρ, and absolute temperature T of an ideal gas:

$$p = \frac{R^*}{m}\rho T \tag{4.1}$$

Here $R^* = 83.13 \times 10^6$ ergs/gm degree $= 1.986$ cal/gm degree, the universal gas constant, and m is the molecular weight of the gas. For actual gases, (4.1) holds as long as they are in a state sufficiently far away from condensation. Therefore, the equation can always be used for the atmospheric gases at ordinary temperatures and pressures, with the exception of water vapor.

For a mixture of two or more gases, as, for instance, for atmospheric air, a similar formula holds. To simplify matters a mixture of only two components will be considered. The gases may have the volumes V_1 and V_2, the masses M_1 and M_2, the same pressure p, and temperature T. Because

$$\rho_1 = \frac{M_1}{V_1} \qquad \text{and} \qquad \rho_2 = \frac{M_2}{V_2}$$

it follows from the gas equation (4.1), as long as the gases are separated in two containers, that

$$p = \frac{R^*}{m_1}\frac{M_1}{V_1}T \qquad \text{and} \qquad p = \frac{R^*}{m_2}\frac{M_2}{V_2}T$$

If the containers are brought together and the separating wall is removed, each gas occupies the whole volume V.

$$V = V_1 + V_2$$

Consequently the sum of the partial pressures of both gases

$$p_1 + p_2 = \frac{R^*}{m_1}\frac{M_1}{V}T + \frac{R^*}{m_2}\frac{M_2}{V}T = p\frac{V_1}{V} + p\frac{V_2}{V} = p$$

This relation states Dalton's law, *viz.*, that the sum of the partial pressures is equal to the total pressure of a mixture of gases. The preceding equation may be written

$$p = \frac{R^*}{m}\frac{M_1 + M_2}{V}T$$

provided that the "molecular weight of the mixture" is defined by

$$\frac{M_1 + M_2}{m} = \frac{M_1}{m_1} + \frac{M_2}{m_2} \tag{4.2}$$

Because $M_1 + M_2 = M$, the total mass of the gas mixture

$$\frac{M_1 + M_2}{V} = \rho$$

Thus, the gas equation for a mixture of gases is also given by (4.1) provided that a mean molecular weight m is introduced according to (4.2). If the mixture consists of more than two components, its molecular weight is given by

$$\frac{\Sigma M_i}{m} = \sum_i \frac{M_i}{m_i} \tag{4.21}$$

From the table in Sec. 3 the molecular weight of the air is found to be $m = 28.97$ if nitrogen, oxygen, argon, and carbon dioxide are taken into account.

Since the universal gas constant R^* appears in the equation mostly divided by the molecular weight m, it will be convenient to introduce the gas constant for (dry) air

$$R = \frac{R^*}{m} = 2.87 \times 10^6 \text{ cm}^2 \text{ sec}^{-2} \text{ (deg)}^{-1}$$

5. Atmospheric Water Vapor. In addition to the other gases enumerated in Sec. 3, atmospheric air contains a certain amount of water vapor which varies widely with time and locality. As long as no condensation or fusion is taking place, water vapor may be treated as an ideal gas. If e is the water-vapor pressure, m_w its molecular weight, $m_w = 18$, ρ_w its density, T its temperature, according to Eq. (4.1)

$$\rho_w = \frac{e}{R^*T} m_w = \frac{m_w}{m} \frac{e}{RT} \tag{5.1}$$

where $m_w/m = 0.621$. It is convenient to introduce the gas constant R for (dry) air in (5.1). The temperature T of the water vapor may be assumed as equal to the temperature of the dry air with which it is mixed. Therefore, it is not necessary to denote it by a subscript w. In meteorology the density of water vapor is frequently called "absolute humidity."

The total density ρ of the moist air is the sum of the density of dry air and of water vapor. The partial pressure of dry air is $p - e$ when p is the total pressure of the moist air.[1] Conseouently

$$\rho = \frac{p - e}{RT} + 0.621 \frac{e}{RT} = \frac{p}{RT}\left(1 - 0.379 \frac{e}{p}\right) \tag{5.2}$$

This equation shows that moist air is lighter than dry air of the same temperature and pressure, for the water vapor is lighter than the air that it replaces.

In problems where only the density of the air is important, dry air of somewhat higher temperature may be assumed to be substituted for the actual moist air. This temperature which the fictitious dry air should have in order to be of the same density as the actual moist air under the same pressure is called the "virtual temperature" T^*. According to (5.2),

$$T^* = \frac{T}{1 - 0.379(e/p)} \tag{5.3}$$

The density of moist air may then be written

$$\rho = \frac{p}{RT^*} \tag{5.4}$$

At a given temperature the water-vapor pressure can rise only up to a certain maximum, the saturation, or maximum, vapor pressure e_m. If the existing water-vapor pressure e is smaller than e_m, evaporation from liquid-water surfaces or ice can take place; if $e = e_m$, an equilibrium is reached between the liquid (or solid) and the gaseous state; if $e > e_m$, condensation[2] occurs. Below the freezing point, one has to distinguish between the saturation pressure over ice and over water.

It should be clearly understood that the fact of saturation is independent of the presence of other gases besides water vapor. If water of a certain temperature is brought into a vessel containing no other gas, the water-vapor pressure, by evaporation, will reach the same saturation value as if air or any other gas

[1] But if all water vapor condenses and falls out as precipitation, the resulting pressure fall will not be e, for the water vapor by itself is not in hydrostatic equilibrium (see Chap. II, Prob. 1).

[2] For modifications of this statement due to the surface tension of water droplets and the pressure of dissolved substances in water, see Sec. 16.

were present. The maximum water-vapor pressure depends only on the vapor temperature. It is, therefore, not strictly correct to say that the air is saturated with water vapor. Some justification for such a statement may, however, be found in the fact that the atmospheric water vapor has the same temperature as the air of which it forms a part. Because the saturation pressure depends on the temperature, its magnitude is indirectly influenced by the air temperature. The expression "saturated air" will therefore be used, for its brevity, in the following discussion.

The variation of e_m with the temperature is given in Table I, (page 341). Tetens[1] has given an empirical formula for e_m based on the laboratory measurements. If e_m is the saturation vapor pressure in millibars and t the temperature in degrees centigrade,

$$e_m = 6.11 \times 10^{\frac{at}{t+b}} \tag{5.5}$$

The constants a and b are as follows:
Over ice,

$$a = 9.5, \qquad b = 265.5$$

Over water,

$$a = 7.5, \qquad b = 237.3$$

A similar theoretical formula can easily be derived from the equation of Clausius-Clapeyron for the heat of condensation.[2]

Besides the absolute humidity, which is used rarely in meteorological practice, the water-vapor content may be expressed by numerous other quantities. The *relative humidity* f is the ratio of the actual vapor pressure to the saturation pressure at the existing temperature,

$$f = \frac{e}{e_m} \tag{5.6}$$

or, according to (5.1),

$$f = \frac{\rho_w}{\rho_{w\,\max}} \tag{5.61}$$

The relative humidity may thus also be defined as the ratio of the actual absolute humidity to the maximum absolute humidity possible at the existing temperature.

[1] Tetens, O., *Z. Geophysik*, **6**, 297, 1930.

[2] See, for instance, D. Brunt, "Physical and Dynamical Meteorology," 2d ed., p. 103, Cambridge University Press, London, 1939.

The *specific humidity* q is the ratio of the absolute humidity (density of water vapor) to the density of the moist air,

$$q = \frac{\rho_w}{\rho} = 0.621 \frac{e}{p - 0.379e} \tag{5.7}$$

The *mixing ratio* w is the ratio of the absolute humidity to the density of dry air,

$$w = \frac{\rho_w}{\rho_{\text{dry air}}} = 0.621 \frac{e}{p - e} \tag{5.8}$$

The following relations exist between the specific humidity and the mixing ratio according to their definitions (5.7) and (5.8):

$$q = \frac{w}{1 + w} \tag{5.71}$$

$$w = \frac{q}{1 - q} \tag{5.81}$$

Because $e \ll p$ as seen from Table I, which gives the maximum water-vapor pressures at different temperatures, (5.7) and (5.8) can in practice be simplified to

$$q \sim w \sim 0.621 \frac{e}{p} \tag{5.82}$$

Mixing ratio and specific humidity are figures without physical dimensions. Owing to their smallness, it is convenient in practice to express them in grams of water vapor per kilogram of air (dry or moist). In Sec. 13, it will be shown that q and w remain constant for dry-adiabatic changes. These quantities are therefore useful for the identification of air masses.

The *dew point* τ is the temperature to which the air has to be cooled, at constant pressure, in order to become saturated.

CHAPTER II

ATMOSPHERIC STATICS. ADIABATIC CHANGES OF DRY AIR

6. The Decrease of the Pressure with Elevation. The atmospheric pressure at any level in the atmosphere represents very accurately the total weight of the air column above the unit area at the level of observation. At greater altitudes the pressure is consequently smaller, for there is less mass above the observer. To find the rate of decrease of the pressure, consider a vertical air column of unit cross section (Fig. 1). At the level z the pressure is p; at the level $z + dz$, it is $p - dp$. The pressure difference is equal to the weight of the air column of the height dz. If dz is chosen sufficiently small so that the density and the acceleration of the gravity g may be regarded as constant in the height interval under consideration,

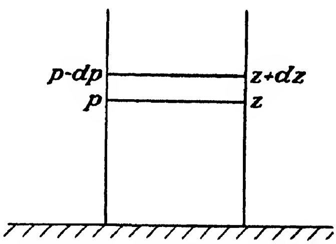

FIG. 1.—Decrease of the pressure with altitude.

$$dp = -g\rho\, dz \tag{6.1}$$

This equation is sometimes called the "hydrostatic equation." As long as the water-vapor content can be neglected, the density

$$\rho = \frac{p}{RT} \tag{4.1}$$

which may be substituted in (6.1). If the variation of g with the altitude is neglected, it follows that

$$\frac{dp}{p} = -\frac{g}{RT}\, dz \tag{6.11}$$

and, by integration, that

$$p = p_0 e^{-\frac{g}{R}\int_0^z \frac{dz}{T}} \tag{6.2}$$

where p_0 is the pressure at the earth's surface. If the temperature is independent of the altitude, (6.2) may be written

$$p = p_0 e^{-\frac{gz}{RT}} \tag{6.21}$$

The assumption of a constant temperature in the vertical direction is a good approximation to the average temperature distribution in the stratosphere. In the lower part of the atmosphere, the troposphere, the temperature distribution is represented better by a function decreasing linearly with the height,

$$T = T_0 - \alpha z$$

The constant α is called the "lapse rate of temperature" or the "vertical temperature gradient," even though the latter expression should rather be reserved for $\partial T/\partial z$. When the temperature increases with the altitude, the lapse rate is negative and the atmosphere shows an "inversion" of the temperature lapse rate; when $\alpha = 0$ within an atmospheric layer, the layer is "isothermal."

If the temperature is a linear function of the height, integration of (6.2) gives the equation

$$p = p_0 \left(\frac{T}{T_0}\right)^{\frac{g}{R\alpha}} \tag{6.22}$$

Upon introducing the geopotential ψ according to (1.3) in (6.1), it follows that

$$dp = -\rho\, d\psi \tag{6.3}$$

Equation (6.3) can be integrated in the same manner as (6.1). When the geopotential is used instead of the geometric height, the variable acceleration of the gravity no longer appears in the equations.

The influence of the atmospheric moisture content on the decrease of the pressure with altitude can be taken into account by using the virtual temperature T^* instead of T. From Eq. (5.3), it followed that moist air of the temperature T and of the vapor pressure e has the same density as dry air of the temperature

$$T^* = \frac{T}{1 - 0.379(e/p)} \tag{5.3}$$

where T^* was the virtual temperature of the air. Therefore,

for moist air the temperature T should be replaced in the preceding equations by the virtual temperature T^*.

7. Height Computation of Aerological Ascents. The barometric formula is used for the solution of a great number of practical problems as, for instance, for the height computations of aerological ascents. Because the aerological data must be

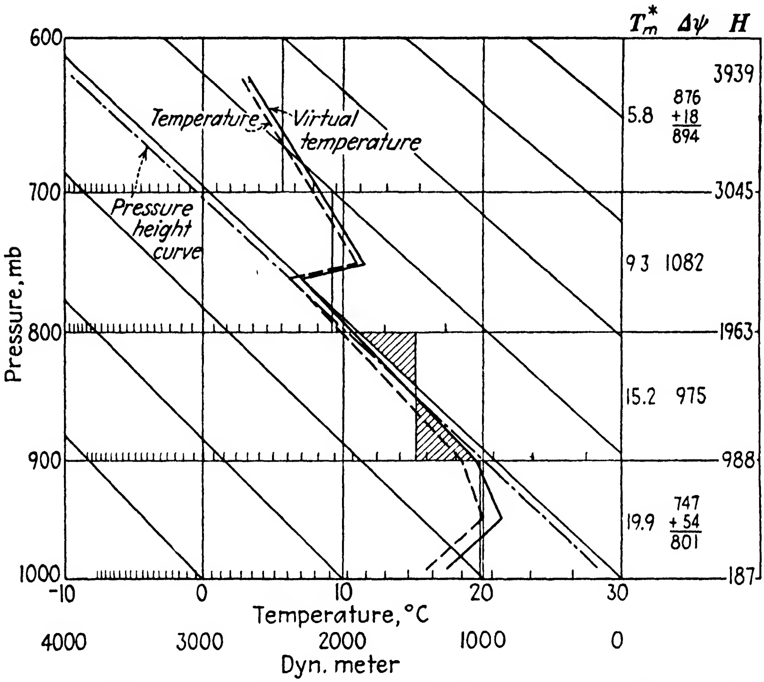

FIG. 2.—Height computation of an aerological ascent after V. Bjerknes. Toronto, July 3, 1939. (The ordinate is $p^{0.288}$, not in p for reasons given on page 23.)

quickly available for the daily weather analysis, a number of methods have been developed for the computation of the height of any point in the atmosphere for which aerological observations are available.[1] Only the method of V. Bjerknes[2] will be described here. From the aerological ascents the pressure p, the temperature T, and the relative humidity f for a number of points in the

[1] STÜVE, G., "Meteorologisches Taschenbuch," II, Akademische Verlagsgesellschaft, Leipzig, 1933.

[2] BJERKNES, V., "Dynamic Meteorology and Hydrography," Chap. VI, Carnegie Institution of Washington, Washington, D. C., 1910.

atmosphere are obtained. They are plotted on a chart whose abscissa is the temperature on a linear scale and whose ordinate is the pressure on a logarithmic scale (T-ln p chart). As an example the airplane ascent made at Toronto on July 3, 1939, is plotted in Fig. 2 (broken curve).

The data for this ascent are

Pressure, mb	990	949	899	760	750	626
Temperature, deg C	15.9	20.0	18.3	6.1	11.0	2.8
Relative humidity, per cent	81	45	56	89	12	27

The height of the Toronto airport is 187 dyn. meters.

To find the height of each point of observation the virtual temperature has to be determined first. Because the computation of this quantity from (5.3) would require, in practice, too much time, provision has been made on the T-ln p chart to obtain it more directly. The difference between virtual temperature and temperature is approximately

$$T^* - T = 0.379 f \frac{e_m(T)}{p} T$$

As long as the relative humidity is 100 per cent, the difference $T^* - T$ is a function of pressure and temperature only. Therefore, the value of $T^* - T$ is fixed by the pressure and temperature of each point on the chart. It is indicated by the distance between each two successive short vertical lines on every isobar representing a multiple of 100 mb. For instance, when the pressure is 700 mb and the temperature +10°C, the virtual temperature of saturated air would be about +12°C. When the relative humidity is less than 100 per cent, $T^* - T$ is obtained by multiplying the difference $T^* - T$ for saturated air by f. In the previous example a relative humidity of 50 per cent would give a virtual temperature of +11°C. In this manner the virtual-temperature curve can be plotted quite easily (full curve in Fig. 2.)

The height may now be expressed in dynamic meters in order to eliminate the acceleration of gravity g. Upon substituting the equation of state for moist air (5.4) in (6.3), it follows that

$$d\psi = -\tfrac{1}{10} R T^* d(\ln p)$$

and, by integration

$$\psi_2 - \psi_1 = -\frac{1}{10} R \int_{\ln p_1}^{\ln p_2} T^* d(\ln p) = \frac{1}{10} R T_{1,2}^* \ln \frac{p_1}{p_2} \tag{7.1}$$

Here

$$(\ln p_1 - \ln p_2) T_{1,2}^* = \int_{\ln p_2}^{\ln p_1} T^* d(\ln p) \tag{7.2}$$

$T_{1,2}^*$ is a suitably defined mean virtual temperature in the layer between p_1 and p_2. $T_{1,2}^*$ can easily be found on the T-ln p chart. Consider, for instance, the virtual-temperature distribution between 900 and 800 mb in Fig. 2. The integral on the right-hand side of (7.2) is represented by the area enclosed between the isobars $p_1 = 900$ mb and $p_2 = 800$ mb and between the isotherm $t = -273°$C (0° abs) and the virtual-temperature curve. Equation (7.2) shows that the isotherm representing the mean virtual temperature T^* must be chosen so that the area enclosed between the isobars 900 mb and 800 mb and the isotherms $-273°$C and $T_{1,2}^*$ is equal to the area given by the integral in (7.2). Thus, the shaded triangles in Fig. 2, which are bounded by the virtual-temperature curve, the isotherm $T_{1,2}^*$, and the isobars 900 mb and 800 mb must be equal. In practice the mean virtual temperature of a layer can be determined quite accurately in this manner even if the virtual-temperature curve is more complicated. The mean virtual temperatures for the ascent at Toronto on July 3, 1939, are given in Fig. 2 under the heading T_m^*.

The dynamic height difference between two pressure levels depends only on the mean virtual temperature of the layer. In practice the height differences between levels whose pressures are multiples of 100 mb, the so-called "standard" isobaric surfaces, are first determined. Tables giving the dynamic height differences between standard isobaric surfaces for various virtual temperatures are available.[1]

If the pressure p_2 is not a standard pressure, (7.1) may be written

$$H_{2,s} = \psi_2 - \psi_s = \frac{1}{10} R T_{s,2} \ln \frac{p_s}{p_2} = \frac{1}{10} R(273° + t^*) \ln \frac{p_s}{p_2}$$

[1] BJERKNES, *op. cit.*, Tables 10*M*–12*M*. LINKE, F., "Meteorologisches Taschenbuch," I, Tables 27 and 27*b*, Leipzig, 1931.

where p_s and ψ_s are the pressure and the dynamic height of the next isobaric surface above or below p_2 and t^* is the mean virtual temperature in degrees centigrade.

Let

$$H_{2,s} = H_0 + \Delta H$$

where

$$H_0 = R \times 273 \ln \frac{p_s}{p_2}$$

and

$$\Delta H = H_0 \frac{t^*}{273}$$

H_0 depends only on p_2 for a given p_s. It gives the dynamic height difference between the two pressures when the mean virtual temperature is 0°C. The correction ΔH depends on the mean virtual temperature and on the dynamic height H_0 at 0°C. H_0 and ΔH can also be obtained from tables.[1] If the temperature t^* is below freezing, ΔH is negative. In the example given in Fig. 2 the uppermost layer extends from 700 mb to 626 mb. The dynamic height difference between these two pressures at $t = 0$°C is 876 dyn. meters. The correction for a mean virtual temperature of 5.8°C is 18 dyn. meters so that the total height difference is 894 dyn. meters. Similarly, it is found that the earth's surface is 802 dyn. meters below the 900-mb surface. Upon adding the height differences (given under the heading $\Delta\psi$ in Fig. 2) successively to the station height the elevations of the various pressure levels are obtained. They are given under the heading H in Fig. 2.

Instead of computing the distance between the surface pressure and the 900-mb surface the distance between the 900-mb surface and the 1000-mb surface might have been computed first and then the distance between the surface of the earth and the 1000-mb surface. To compute the latter the virtual-temperature curve has to be extrapolated downward, for the 1000-mb surface is below the ground in the present example. As long as the surface pressure is not much lower than 1000 mb, this extrapolation will not give rise to an appreciable error.

After the height of the standard isobaric surfaces has been computed, a curve may be drawn representing the pressure distribution with height, the pressure-height curve. When the

[1] Bjerknes, V., *op. cit.*, Table 9*M*. Linke, *op. cit.*, Table 28.

linear abscissa is used as the height scale, the pressure-height curve must be approximately a straight line. This permits the detection of major errors in the computations. The heights of intermediate pressure levels can be obtained from the pressure-height curve with an accuracy that is sufficient for most meteorological purposes.

8. Adiabatic Changes of Dry Air. It is known from physics that heat is a special form of energy which may be changed into other forms such as mechanical work, for instance. The unit of heat is the calorie, or more precisely the gram-calorie. It is the amount of heat required to raise the temperature of 1 gram of water 1°C. This amount varies somewhat with the temperature and the pressure so that the gram-calorie is defined more accurately as the amount of heat necessary to raise the temperature of 1 gram of water from 14.5°C to 15.5°C at the normal pressure of 760 mm Hg. But for meteorology this refinement is not important. The gram-calorie represents a certain amount of mechanical energy. Experiments have shown that it is equal to 4.185×10^7 ergs. Therefore, in order to convert calories into ergs, one has to multiply by a factor

$$J = 4.185 \times 10^7 \text{ ergs/cal}$$

J is called the *mechanical equivalent of heat.*

If the amount of heat dq is required to raise 1 gram of a substance through a change in temperature dT, the specific heat

$$c = \frac{dq}{dT}$$

In the case of gases, one has to distinguish among various specific heats according to the change of state that the gas undergoes. If its specific volume is kept constant, the specific heat at constant volume c_v has to be considered. If the pressure remains constant the specific heat at constant pressure c_p appears. For air, $c_v = 0.170$ cal/gm degree.

The first law of thermodynamics[1] states that an amount of heat dq added to a gas is used partly to increase the internal energy and partly to do work against the outer pressure by expansion of the gas. In the case of an ideal gas the internal energy depends

[1] A derivation of the first law for ideal gases will be found in any textbook on thermodynamics.

only on the temperature, and the change of the internal energy is given by $c_v\, dT$. To find the work done by expansion against an outer pressure p, consider a cylinder with a movable piston of cross section F (Fig. 3). The force exerted by the outer pressure on the piston is pF. When a gas inside the cylinder expands and moves the piston a distance dx, the work done by the gas is

$$pF\, dx = p\, dV$$

where dV is the change of the gas volume. Consequently the amount of work done by the unit mass is $p\, dv$ where v is the specific volume. If the work is to be expressed in thermal units, it has to be multiplied by $A = 2.390 \times 10^{-8}$ cal/erg, the reciprocal of the mechanical equivalent of heat.

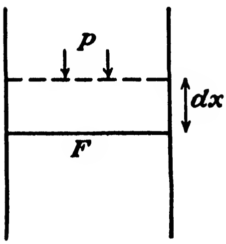

FIG. 3.—Computation of work done by expansion.

The first law of thermodynamics applied to the unit mass of an ideal gas may, therefore, be written

$$dq = c_v\, dT + Ap\, dv \tag{8.1}$$

Dry air may be regarded as an ideal gas at the temperatures occurring in the atmosphere; but the water vapor requires a separate treatment, for it condenses and freezes at atmospheric temperatures.

Upon substituting here the gas equation (4.1), Eq. (8.1) changes into

$$dq = (c_v + AR)\, dT - \frac{ART}{p}\, dp \tag{8.2}$$

If the pressure is kept constant and only the temperature changes, the specific heat at constant pressure is obtained. It follows that

$$c_p = \left(\frac{\partial q}{\partial T}\right)_p = c_v + AR \tag{8.3}$$

For dry air the specific heat at constant pressure is 0.239. Actually, the two specific heats of the air are variable, but the variation is so small that it may be neglected.

Frequently, it can be assumed that during atmospheric processes the heat content of the air under consideration remains

unchanged, *i.e.*, that

$$dq = 0$$

Such variations are called "adiabatic." Strictly speaking, it has also to be assumed that the variations of the gas are infinitely slow so that the whole finite change of the gas consists in a succession of equilibrium states, as is discussed in the textbooks on thermodynamics (see page 59). Changes of state in the atmosphere may frequently be assumed to resemble adiabatic processes, for the loss or gain of heat by radiation or conduction is often small compared with the change caused by compression or expansion, especially in connection with vertical motion.

For adiabatic changes, it follows from (8.2) and (8.3) that

$$\frac{dT}{T} = \frac{AR}{c_p}\frac{dp}{p} \tag{8.4}$$

or, by integration, that

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\kappa} \tag{8.41}$$

where

$$\kappa = \frac{AR}{c_p} = \frac{c_p - c_v}{c_p} = 0.288$$

T_0 is the temperature at the pressure p_0. Upon introducing the gas equation (4.1) into (8.41), a relation between pressure and density is obtained,

$$p\rho^{-\lambda} = \text{const} \tag{8.42}$$

where

$$\lambda = \frac{c_p}{c_v} = 1.405$$

If the specific volume and temperature are introduced in (8.42) for the pressure and the density, the adiabatic relation may be written

$$Tv^{\lambda-1} = \text{const} \tag{8.43}$$

For the theoretical considerations, it is sometimes useful to generalize the adiabatic condition $dq = 0$ by assuming instead that

$$dq = c\,dT$$

where c is a constant of the dimensions of a specific heat. Changes of state of a gas that follows this more general condition

are called "polytropic." The relation between pressure and temperature now becomes

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\bar{\kappa}} \tag{8.5}$$

where

$$\bar{\kappa} = \frac{c_p - c_v}{c_p - c}$$

and the relation between pressure and density becomes

$$p\rho^{-\bar{\lambda}} = \text{const} \tag{8.51}$$

where $\bar{\lambda} = \dfrac{c_p - c}{c_v - c}$ is the "modulus" that characterizes each polytropic curve. The following polytropic curves are of special interest:

Isobaric curves,

$$\bar{\lambda} = 0, \qquad c = c_p$$

Isothermal curves,

$$\bar{\lambda} = 1, \qquad c = \pm\infty$$

Isosteric curves,

$$\bar{\lambda} = \infty, \qquad c = c_v$$

Adiabatic curves

$$\bar{\lambda} = \frac{c_p}{c_v}, \qquad c = 0$$

9. Potential Temperature. The Dry-adiabatic Lapse Rate. Vertical Stability of Dry Air. The potential temperature Θ of dry air is the temperature that the air would assume if brought adiabatically from its actual pressure to a standard pressure P that is generally chosen equal to 1000 mb. Consequently

$$\Theta = T\left(\frac{P}{p}\right)^{\kappa} \tag{9.1}$$

When dry air undergoes adiabatic changes, the potential temperature remains constant. It is a conservative property of dry air, conservative with respect to adiabatic changes.

A unit of air that moves vertically upward or downward expands or contracts because the pressure exerted on it by the surrounding atmosphere decreases or increases. The effects of vertical motions in the atmosphere are as a rule so marked that the influence of radiation and convection may be neglected

and the motion can be assumed to be adiabatic. Then it follows from (8.4) that the temperature variation of the ascending air

$$\frac{1}{T}\frac{\partial T}{\partial z} = \kappa \frac{1}{p}\frac{\partial p}{\partial z}$$

The decrease of the pressure with the altitude $\partial p/\partial z$ depends not on the temperature T of the moving air but on the temperature T' of the surrounding air, so that, with (6.11),

$$\frac{\partial T}{\partial z} = -\frac{g\kappa}{R}\frac{T}{T'} = -\frac{gA}{c_p}\frac{T}{T'} \tag{9.2}$$

Because the ratio T/T' is mostly not very different from unity, the decrease of the temperature of the ascending air is very closely given by

$$\Gamma = \frac{g\kappa}{R} = 0.98°\text{C} \times 10^{-4}/\text{cm} = 0.98°\text{C}/100\text{ m} \tag{9.21}$$

This quantity is called the "adiabatic lapse rate" or, more accurately, the "dry-adiabatic lapse rate."

If the lapse rate of the air is smaller than the adiabatic lapse rate, the air is in "stable" equilibrium. To show this condition, consider a parcel of air that originally had the temperature of its surroundings.[1] It will be assumed that the motion of such a parcel of air does not disturb the stratification of the environment, an assumption that is obviously not fully justified if the motion of the parcel is to be studied.[2] Such a simplification is, however, permissible when, as in this case, the static problem of the stability or instability of the atmosphere is being investigated. If the parcel is lifted, it cools at the adiabatic rate while the temperature of the surrounding air decreases at a rate less than the adiabatic. Thus, the displaced air parcel attains at its new position a temperature lower than the temperature of the surrounding air. Because the pressures of the displaced air and of the surrounding air are the same, the density and weight of the displaced air are greater owing to its lower temperature. Therefore, it sinks back to its original position. The argument

[1] For a rigorous proof, see H. Ertel, "Methoden und Probleme der dynamischen Meteorologie," p. 57, Verlag J. Springer, Berlin, 1938.

[2] Bjerknes, J., *Quart. J. Roy. Met. Soc.*, **64**, 325, 1938. Petterssen, S., *Geofys. Pub.*, 12, No. 9, 1939; "Weather Analysis and Forecasting," p. 64, McGraw-Hill Book Company, Inc., New York, 1940.

obviously holds also when downward motion takes place, for the moving parcel of air now becomes warmer than the air surrounding it and is therefore lighter and returns upward to its original position.

If the lapse rate of the air is larger than the adiabatic, on the other hand, a parcel of air moving upward would arrive in its new position warmer and lighter than the surrounding air and would continue to ascend. The air is in "unstable" equilibrium. It should be understood that even in this case vertical motion will not start spontaneously. As long as the mass distribution of the atmosphere is undisturbed, equilibrium prevails. Lighter air is above heavier air. Only after an initial disturbance has been brought about will strong vertical motions occur, and in the case of a less-than-adiabatic lapse rate such a disturbance is damped by the stable stratification of the air.

If the lapse rate happens to be adiabatic, a particle moved up or down has always the same temperature as the surrounding air and is thus always in equilibrium. The equilibrium is "indifferent." In general the lapse rate of temperature in the atmosphere is below the adiabatic, about 0.6°C/100 km.

If the lapse rate is called α, the equilibrium conditions for dry air may be written in the form

$$\begin{aligned} \alpha &< \Gamma, \text{ stable equilibrium} \\ \alpha &= \Gamma, \text{ indifferent equilibrium} \\ \alpha &> \Gamma, \text{ unstable equilibrium} \end{aligned}$$

Upon differentiating (9.1) logarithmically and substituting from (6.11), it is seen that the lapse rate of potential temperature is related to the lapse rate of temperature by

$$\frac{\partial \Theta}{\partial z} = \frac{\Theta}{T}(\Gamma - \alpha) \tag{9.3}$$

This formula permits one to express the equilibrium conditions for dry air by the lapse rate of potential temperature. Dry air is in stable equilibrium when $\partial\Theta/\partial z > 0$, in indifferent equilibrium when $\partial\Theta/\partial z = 0$, and in unstable equilibrium when $\partial\Theta/\partial z < 0$. Near the surface of the earth where the pressure is not very different from 1000 mb, Θ is approximately equal to T. Thus, approximately,

$$\Theta = T + \Gamma z \tag{9.31}$$

If lines of equal potential temperature are drawn on a pressure-temperature chart as used in Sec. 7 (Fig. 2), it can be seen at a glance whether the (dry) air is in stable equilibrium or not. If the inclination of the ascent curve is steeper than the inclination of the lines of constant potential temperature, the stratification is stable; if it is less steep, unstable. Where the inclination is the same for both curves, indifferent equilibrium exists. The lines of constant potential temperature are also called *adiabats*, for the point representing the pressure and temperature of a mass of dry air moving adiabatically remains on the same line. The whole chart is also referred to as an *adiabatic chart.*

On a chart whose abscissa is the temperature and whose ordinate is the logarithm of the pressure the adiabats are obviously not straight lines. If the ordinate were p^κ, however, the adiabats would become straight lines as is seen from (9.1). Because this is convenient for many problems, adiabatic charts with p^κ as ordinate have also been constructed. On such charts the determination of the mean temperature outlined in Sec. 7 is not quite exact; but the error involved is negligible, especially when the pressure intervals are chosen not larger than 100 mb.

The previous considerations concerning the stability of dry air undergoing adiabatic changes can be extended to polytropic changes in general. If the atmosphere follows a polytropic law characterized by a modulus $\bar{\lambda}$, it is in stable equilibrium provided that the temperature lapse rate is smaller than

$$\frac{g\bar{\kappa}}{R} = \frac{g}{R}\frac{\bar{\lambda} - 1}{\bar{\lambda}} \tag{9.4}$$

This expression plays for an atmosphere following such a polytropic law the same role as the adiabatic lapse rate for an atmosphere following an adiabatic law of compression and expansion.

10. The Influence of Vertical Motion on the Temperature Lapse Rate and on the Stability of Dry Air. When large-scale vertical motions take place in the atmosphere, whole layers of air may be moved up or down so that they are brought under different pressure and their horizontal cross section is changed. In anticyclones, for example, a descending motion frequently takes place under simultaneous spreading of the air layers, as indicated schematically in Fig. 4, while the relative position of the layers remains the same. Under these circumstances the

lapse rate of temperature changes owing to adiabatic expansion.[1] The change in lapse rate can be computed quite easily under the simplifying assumption that the height of the layer considered is small. The cross section of the layer (Fig. 5) may be A before the change has taken place, Θ and $\Theta + d\Theta$ the potential temperatures at the lower and upper surfaces, ρ its density, and dz its height. After the change, let the cross section be A', the density ρ', the height dz'; the potential temperature is not altered as long as the process is adiabatic. Because the mass of the layer remains constant,

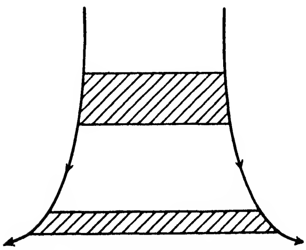

FIG. 4.—Subsiding motion and shrinking.

$$A\rho\, dz' = A'\rho'\, dz'$$

Here it is assumed that the height of the layer is small so that ρ and ρ' may be regarded as independent of the height within the layer. Owing to the invariance of the potential temperature,

$$\frac{d\Theta}{dz'} = \frac{d\Theta}{dz}\frac{dz}{dz'} = \frac{d\Theta}{dz}\frac{A'\rho'}{A\rho}$$

When the density is expressed by the pressure and temperature

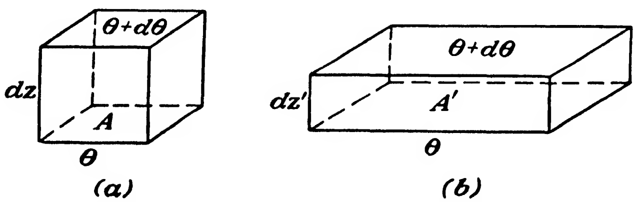

FIG. 5.—Variation of the lapse rate of temperature due to vertical motion.

according to the gas equation (4.1), this equation may be written

$$\frac{d\Theta}{dz'} = \frac{d\Theta}{dz}\frac{A'}{A}\frac{p'}{p}\frac{T}{T'} \tag{10.1}$$

The lapse rate of potential temperature can be expressed by the lapse rate of temperature according to (9.3),

$$\frac{\Theta}{T'}(\Gamma - \alpha') = \frac{\Theta}{T}(\Gamma - \alpha)\frac{A'}{A}\frac{p'}{p}\frac{T}{T'}$$

[1] SCHMIDT, W., *Beitr. Phys. Atm.*, **7**, 103, 1917.

Thus

$$\alpha' = \Gamma - \frac{A'}{A}\frac{p'}{p}(\Gamma - \alpha) \tag{10.2}$$

An increase in cross section acts in the same way as an increase in pressure, *i.e.*, as a sinking of the layer of air; a decrease in cross section like a decrease in pressure, *i.e.*, as a lifting. For the rest of the discussion, it will therefore be assumed for the sake of simplicity that A remains unchanged even though the change in cross section may actually be as important as the pressure change. Equation (10.2) then becomes

$$\alpha' = \alpha - (\Gamma - \alpha)\frac{\Delta p}{p} \tag{10.21}$$

where

$$\Delta p = p' - p$$

First the case where $\Gamma > \alpha$ may be considered, *i.e.*, where the temperature distribution is stable originally. When $\Delta p > 0$, *i.e.*, when the air descends, $\alpha' < \alpha$, *i.e.*, the lapse rate becomes smaller, and when Δp is sufficiently large, α' may even become zero or negative. When the lapse rate is negative, the temperature increases with elevation. Thus an inversion may be formed by sinking and spreading of the air. This process occurs frequently in the center of stagnant anticyclones where large inversions are observed which from their origin are called *subsidence inversions.*[1] Because the stratification is more stable the smaller the lapse rate is compared with the adiabatic lapse rate, the result may also be stated by saying that descending motion in an atmosphere with originally stable stratification increases the stability of the air. On the other hand, when the air ascends (or when its cross section decreases), *i.e.*, when $\Delta p < 0$, the lapse rate α' becomes larger.

In the rare case of originally unstable stratification, $\alpha > \Gamma$, the effect of upward and downward motions is just the opposite. Downward motion increases the lapse rate; upward motions make the lapse rate smaller.

When the lapse rate was originally adiabatic ($\Gamma = \alpha$) it remains unchanged. The method can also be extended to air columns of finite height,[2] but the lapse rates resulting from vertical

[1] NAMIAS, J., *Harvard Met. Studies*, No. 2, 1934.

[2] HAURWITZ, B., *Ann. Hydr.*, **59**, 22, 1931.

adiabatic motion of finite air columns are not very different from those obtained from the preceding formula (10.21).

For a graphical determination of the change of the lapse rate in a layer of air that ascends or descends adiabatically, the adiabatic chart may be used. The full curve AB in Fig. 6 may represent the original pressure and temperature distribution. When the layer is subjected to vertical adiabatic motion, each point of AB must move along an adiabat (broken curves). If A comes to rest at a pressure p_C, and B at p_D without a disturbance of the relative position of the points along AB, the line CD represents the new stratification of the layer. It may be noted that owing to the conservation of mass the pressure difference

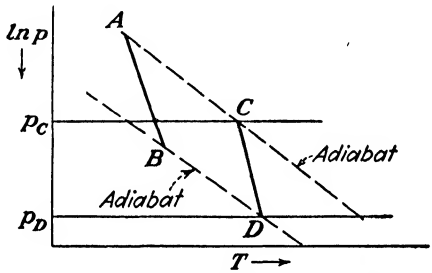

Fig. 6.—Change of the lapse rate in a subsiding layer.

between C and D is equal to the original pressure difference between A and B if the cross section of the layer remains unchanged. If the cross section changes, the pressure difference changes also. The mass remains constant, so that

$$(p_B - p_A)A = (p_D - p_C)A'$$

11. The Relation between Pressure and Temperature Variations. Because the surface pressure is the total weight of the air column above the point of observation, it varies when the mass of air changes at any level.

It may be assumed that the atmosphere is divided into n layers of equal height h. If the height h is chosen not too large, the temperature in each layer may be regarded as constant. If the surface pressure is p_0, the pressure at the highest point of observation nh is p_n and the mean temperature in the layer k from $(k - 1)h$ to kh is T_k, it follows by repeated application of the barometric formula (6.21) that

$$p_0 = p_n e^{\frac{g}{R}h \sum_{k=1}^{n} \frac{1}{T_k}} \tag{11.1}$$

This formula expresses the surface pressure by the pressure at the highest point of observation and by the temperature in the intermediate layers. Upon differentiating logarithmically with respect to time, (11.1) becomes

$$\frac{\partial p_0}{\partial t} = \frac{p_0}{p_n}\frac{\partial p_n}{\partial t} - \frac{g}{R} h p_0 \sum_{k=1}^{n} \frac{1}{T_k^2}\frac{\partial T_k}{\partial t} \tag{11.2}$$

The differential quotients with respect to the time may be identified with the 12-hourly or 24-hourly variations of pressure and temperature which can be determined by successive aerological ascents. Formula (11.2) is, of course, correct in so far as it gives the changes that must occur simultaneously in the different atmospheric layers if static equilibrium persists. But it has frequently led to misinterpretations that will be discussed here, for such a discussion will clarify the mechanism that links the pressure and temperature variations at various levels in the atmosphere.[1]

When the temperature in the intermediate layers remains unchanged, (11.2) is reduced to

$$\frac{\partial p_0}{\partial t} = \frac{p_0}{p_n}\frac{\partial p_n}{\partial t} \tag{11.21}$$

The pressure variation at the ground is p_0/p_n times larger than the simultaneous pressure variation at the level nh. For instance, when in the atmosphere at an altitude where the average pressure is 250 mb a pressure variation of 5 mb is observed, the simultaneous pressure variation at the ground where the average pressure is 1000 mb should be 20 mb. From (11.21), it has been inferred that the variations of the pressure at greater heights dominate the surface pressure variations completely, especially after it was discovered from aerological ascents that the daily variations of the pressure at greater altitude are of about the same magnitude as at the ground. Obviously, however, (11.21) cannot mean that a pressure change dp_n at a height where the pressure is p_n causes a surface pressure change $\frac{p_0}{p_n}\, dp_n$. This

[1] Haurwitz, B., and Haurwitz, E., *Harvard Met. Studies*, No. 3, 1939.

cannot be correct, for an increase of the pressure dp_n shows only that above the level of the pressure rise the mass increases by dp_n/g per unit cross section. As long as no advection takes place in the lower layers, the mass of the air column increases only by dp_n/g and not by $\frac{p_0}{p_n}\frac{dp_n}{g}$.

To interpret the meaning of (11.21) correctly, it has to be considered from a different angle. Suppose that the pressure at the earth's surface changes by dp_0, which may be assumed positive in order to have a more concrete picture. This pressure rise indicates that the total mass of the air column per unit cross section has increased by dp_0/g. If the advection of mass has taken place at and above an altitude H, the air column below H is compressed by the added air and a part of the air previously above H sinks below this level. The change in pressure dp_n at this level H is therefore equal to the weight of the air added at and above H minus the weight of the air that sinks below H owing to the compression. The increase of the mass above H, dp_n/g, is thus only a fraction of the total mass that has been added above H. Consequently the pressure increase at the surface must be larger than dp_n because the variation of the surface pressure represents the total mass added to the air column. Thus Eq. (11.21) actually states, *not* that a pressure variation dp_n at the height where the pressure is p_n causes a surface pressure variation p_0/p_n times larger, but that the total pressure variation dp_0 at the level H is diminished proportional to the ratio of the pressure p_n/p_0 owing to the compressibility of the atmosphere.

Another misinterpretation of (11.21) concerns the second term on the right-hand side containing the temperature variations. If the temperature changes only in the layer k, from $(k - 1)h$ to kh the surface pressure change, according to (11.21), is

$$\frac{\partial p_0}{\partial t} = -\frac{g}{R}h\frac{p_0}{T_k^2}\frac{\partial T_k}{\partial t} \tag{11.22}$$

This equation shows that for one and the same temperature change at different heights the surface pressure variation is usually somewhat larger for the temperature changes at higher altitudes because the temperature usually decreases with height. However, one and the same temperature variation at a higher altitude affects a smaller mass when layers of the same thickness

are considered, on account of the decrease of the air density with altitude. Therefore, the change of the total mass of an air column should be smaller, the higher the layer in which the temperature variation takes place. To explain this apparent paradox, consider the variation of the pressure dp_{k-1} at the bottom of the layer k due to a change of the temperature in the layer k, while the pressure p_k at the top of the layer remains constant. This variation is expressed by

$$\frac{\partial p_{k-1}}{\partial t} = -\frac{g}{R} h \frac{p_{k-1}}{T_k^2} \frac{\partial T_k}{\partial t} \tag{11.23}$$

According to this expression the pressure variation at the bottom of any layer due to the temperature variation within the layer is proportional to the pressure at the bottom of the layer. Consequently, for a layer at greater height the pressure variation is smaller even though the temperature variation be the same. But according to (11.21) the surface pressure variation connected with the variation is given by

$$\frac{\partial p_0}{\partial t} = \frac{p_0}{p_{k-1}} \frac{\partial p_{k-1}}{\partial t}$$

This equation combined with (11.23) leads back to (11.22). This rather roundabout derivation of (11.22) shows that the change in weight of an air column due to the temperature variation of a given layer is proportional to the pressure at the lower boundary of this layer. Thus, the effect of a given temperature change in a layer of given thickness upon the weight of the whole air column is actually smaller the higher up in the atmosphere this layer is located, as should be expected considering that less mass is affected by the temperature variation if it occurs at a higher altitude. But if equilibrium in the underlying air column is to be maintained, its mass must also be changed to such an extent that the surface pressure varies according to (11.22).

12. Computation of the Advection at Great Heights. In reality the connection between pressure and temperature variations at different altitudes will be more complicated than was assumed in Sec. 11. In general the variations in the underlying layers are not of precisely the right order of magnitude to balance the change of the weight of the layer k. Therefore the air that is in this layer, between the heights $(k - 1)h$ and kh, does not remain there because the underlying air column is com-

pressed. The vertical motions that are then taking place are indicated schematically in Fig. 7. It is possible to observe only the "local"[1] temperature variation between the levels $(k-1)h$ and kh that is in the layer $ABCD$ before and $A'B'C'D'$ after the advection of air of different temperature. This is the temperature variation that is inserted in (11.22) and (11.23). The "individual"[1] temperature variation that actually accounts for the change in mass due to advection is the temperature variation of the air which was at $ABCD$ before the advection and which is at $EFGH$ after the advection.

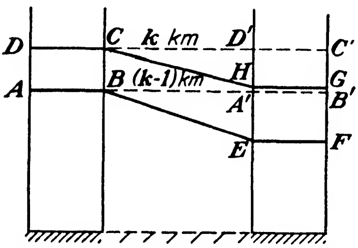

FIG. 7.—Effect of cooling on the position of a layer of air.

The misinterpretation of (11.22) was thus obviously due to a confusion between local and individual temperature variation. It follows that Eq. (11.2), which was derived from the barometric formula, is unsuitable for giving an indication of the effects of advection on the change of the surface pressure. The individual pressure variation $\delta\pi$ at a surface that consists always of the same air particles and that moves up or down when the weight of the overlying air changes would be a more satisfactory expression of the total advection above this fixed level. $\delta\pi$ is zero when no advection takes place or when the air is replaced by air of the same density so that the advection would appear to be zero. However, since in this case the advection is without effect on the surface pressure, it is without interest and the failure of $\delta\pi$ to measure the advection in this case is not serious. The quantity $\delta\pi$ may be called the "advection function." It was first discussed by Rossby,[2] later by Ertel and Sjan-zsi-Li.[3]

The complete treatment of this problem cannot be given here. Instead a somewhat simpler problem, a typical example of the way in which problems of this kind may be attacked,[4] will be discussed here.

[1] The local variation is that observed at a fixed place while the individual variation is that observed following one and the same particle of air.

[2] ROSSBY, C.-G., *Beitr. Phys. Atm.*, **13**, 163, 1927 and **15**, 240, 1928.

[3] ERTEL, H., and SJAN-ZSI-LI, *Z. Physik*, **94**, 662, 1935.

[4] ROSSBY, C.-G., *loc. cit.*, 1927.

It may be assumed that the advection takes place only at a great height. The problem is to find the temperature and pressure changes below this level as functions of the altitude. When these variations are determined, it can be decided by comparison with the observed variations whether the assumption is correct that the advection occurred at a great height only.

The variation of the surface pressure δp_0 is equal to the total advection of mass at great heights, expressed by $\delta\pi$. Below the level of advection the individual pressure change is also equal to $\delta\pi$,

$$\delta_i p = \delta\pi$$

The operator δ stands for a small change of the quantity with time. The subscript i refers to the individual change, and l indicates local changes. The local pressure change is equal to the sum of the individual pressure change and the effect of the vertical displacement of the air

$$\delta_l p = \delta_i p + g\rho\,\delta z = \delta\pi + g\rho\,\delta z \tag{12.1}$$

where δz is the height variation of the surface on which the individual pressure change is measured. The condition of continuity states that the mass $\rho\,dz$ of the layer whose thickness was originally dz is constant, *i.e.*, that

$$\delta(\rho\,dz) = 0$$

or

$$\frac{\delta_i\rho}{\rho} + \frac{\delta(dz)}{dz} = 0$$

For brevity, no change of the cross section is assumed. From the adiabatic equation (8.42),

$$\frac{\delta p}{p} - \lambda\frac{\delta_i\rho}{\rho} = 0$$

Because the adiabatic relation refers to individual particles, δp is here the individual pressure change $\delta\pi$. From geometric considerations, $d(\delta z) = \delta(dz)$. Upon combining the three preceding equations, it follows that

$$\frac{d(\delta z)}{dz} = -\frac{1}{\lambda}\frac{\delta\pi}{p}$$

By integration,

$$\delta z = -\frac{\delta\pi}{\lambda}\int_0^z \frac{dz}{p} \tag{12.2}$$

where the condition

$$(\delta z)_{z=0} = 0$$

has been introduced.

In practice the observed temperature distribution can frequently be replaced with sufficient accuracy by a mean temperature T_m when the vertical pressure distribution is to be computed. According to (6.21), it is then

$$p = p_0 e^{-\frac{g}{R}\frac{z}{T_m}}$$

With this relation, (12.2) assumes the simpler form

$$\delta z = -\frac{RT_m}{gp_0}\left(\frac{p_0}{p} - 1\right)\frac{\delta\pi}{\lambda} \tag{12.21}$$

The local pressure change, from (12.1) and (12.2), is

$$\delta_l p = \delta\pi\left(1 - \frac{g}{\lambda}\frac{p}{RT}\int_0^z \frac{dz}{p}\right) \tag{12.3}$$

If the approximate expression (12.21) is used,

$$\delta_l p = \delta\pi\left[1 - \frac{1}{\lambda}\frac{T_m}{T}\left(1 - \frac{p}{p_0}\right)\right] \tag{12.31}$$

The local temperature change due to advection at great heights can be expressed by

$$\delta_l T = \delta_i T - \frac{dT}{dz}\,\delta z \tag{12.4}$$

which is analogous to (12.1) for the pressure. Because according to the adiabatic relation (8.4) the individual temperature change is connected with the individual pressure change by

$$\frac{\delta_i T}{T} = \kappa\,\frac{\delta\pi}{p}$$

the local temperature change, upon putting $\alpha = -\frac{dT}{dz}$, is

$$\delta_l T = \kappa\frac{T}{p}\,\delta\pi\left(1 - \frac{\alpha}{\lambda - 1}\frac{p}{T}\int_0^z \frac{dz}{p}\right) \tag{12.5}$$

With the simplifying assumption leading to (12.21),

$$\delta_l T = \kappa\frac{T}{p}\,\delta\pi\left[1 - \frac{\alpha}{\lambda - 1}\frac{R}{g}\frac{T_m}{T}\left(1 - \frac{p}{p_0}\right)\right] \tag{12.51}$$

With the aid of (12.2), (12.3), and (12.5) the vertical displacement of a particle and the pressure and temperature variation at a given altitude due to advection at a higher level can be computed when the original pressure and temperature distribution in the atmosphere are known. The generalization for arbitrary polytropic changes is obvious.

Equation (12.2) or, more clearly, (12.21) shows that the altitude of a particle of air decreases when mass is added and increases when air is taken away at great heights, the changes in height being greater at greater elevations. The local pressure change is largest at the earth's surface where it is, of course, equal to the total advection and decreases with altitude, as is seen from (12.3) and (12.31).

Rossby[1] has given an example that shows very clearly advection at great heights over Trappes (France) on Apr. 11 and 13, 1912. The results of his analysis are reproduced in the following table. The first five columns contain the height, the pressure

Height, km	Pressure, mb, Apr. 11, 1912	Temperature, deg. abs, Apr. 11, 1912	$\delta_t p$, mb	$\delta_t T$, deg C	$-\delta z$, m	$\delta_t \pi$, mb
0 17	990	278 0	+19	− 5 6	0	+19
1	894	274 1	+15	− 2 6	10	16
2	788	268.8	+13	− 0 1	24	16
3	693	262.2	+12	+ 4 5	39	16
4	605	259.6	+11	+ 0.7	57	15
5	533	253.9	+11	+ 0 5	77	16
6	465	246.3	+ 9	+ 0.7	100	16
7	404	239.9	+ 8	+ 0.9	127	15
8	349	233 5	+ 8	+ 0.4	157	16
9	301	225.7	+ 7	+ 0 1	193	15
10	257	217 5	+ 5	+ 3 4	234	13
11	220	208 9	+ 7	+11.7	282	19
12	187	206.7	+ 7	+13 9	338	19

and temperature observed at this height on the first day, and the change of pressure and temperature at these heights from th eleventh to the thirteenth. The pressure ... at al whereas the temperature falls only in the l

[1] Rossby, C.-G., *loc. cit.*, 1927.

only this layer can contribute directly to the increase of the surface pressure. In the upper troposphere the temperature variation is very small and positive; above 10 km the temperature has increased considerably. It may therefore be expected already from the observed data that the advection has mainly taken place at great altitudes. In order to show that this is the case, $\delta\pi$ has been computed from (12.3). If $\delta\pi$ were variable, it would show where advection has occurred. Actually, it has approximately the same value of 16 mb at all heights for which observations are available. The pressures observed during the ascent are given only in whole millimeters of mercury so that for this reason alone errors of 1 mb are possible in $\delta_l p$. Thus, the great increase of the surface pressure appears to be mainly due to advection in the stratosphere, whereas the surface layers up to 2 km contribute slightly, owing to the advection of a cold air mass. The variation of the height of an individual particle of air is more than 300 m at 12 km, the highest point for which observations are available. Because mass has been added to the air column, the height variation is negative; each layer of air has been pressed down by the advection. Rossby has also computed the local temperature change from (12.5). The agreement with the observed changes is, however, not very good. This may be due to advection in the lower layer which is large enough to prevent agreement between observed and computed temperature changes though it is not large enough to have a noticeable effect on the pressure variation.

The considerations leading to (12.3) have been generalized by Rossby[1] and Ertel and Sjan-zsi-Li[2] to allow for advection in the lower part of the air column. Ertel and Li have shown that the formula (12.3) holds also in this more general case.

In the preceding discussion the possible effects of a change of the cross section are neglected. Van Mieghem[3] has shown that these changes are also quite important and have to be taken into account in a more complete analysis.

[1] Rossby, C.-G., *loc. cit.*, 1928.

[2] Ertel, H., and Sjan-zsi-Li, *loc. cit.*, 1935. See also H. Ertel, "Methoobleme d lynamischen Meteorologie," p. 80, Verlag J. Springer,

'. *roy. de Belg. bull. class. sci.*, 5th ser., **25**, 243,

and hence

$$\frac{dp}{p} = \frac{de}{e} \tag{13.2}$$

Therefore,

$$dq = c_p\, dT - \frac{ART}{p}\, dp \tag{13.3}$$

The same relation for w gm[1] of unsaturated water vapor is

$$w\, dq = wc_{pw}\, dT - AR\frac{m}{m_w}\, wT\frac{de}{e}$$

Here $c_{pw} = 0.466$, the specific heat at constant pressure of water vapor, and m/m_w is the ratio of the molecular weights of dry air and water vapor.

According to (13.2),

$$w\, dq = wc_{pw}\, dT - AR\frac{m}{m_w}\, wT\frac{dp}{p} \tag{13.4}$$

By addition of (13.3) and (13.4),

$$(1 + w)\, dq = (c_p + wc_{pw})\, dT - AR\left(1 + \frac{m}{m_w}\, w\right) T\frac{dp}{p}$$

We may introduce

$$c_p{}^* = c_p\,\frac{1 + (c_{pw}/c_p)w}{1 + (m/m_w)w} = c_p\,\frac{1 + 1.95w}{1 + 1.609w} \tag{13.41}$$

as a specific heat at constant pressure for unsaturated moist air. Because w is small, of the order of magnitude 10^{-2}, $c_p{}^*$ differs very little from c_p. For adiabatic changes, because $dq = 0$,

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{AR}{c_p{}^*}} \tag{13.5}$$

This equation is for all practical purposes identical with the equation for adiabatic changes of dry air (8.41).

In dealing with moist air, it is frequently advantageous to use the partial potential temperature[2] which is defined as the temperature which the air would assume if it were brought adiabatically from the actual partial pressure of dry air $p - e$ to the standard pressure $P = 1000$ mb. According to (13.5)

[1] This is obviously the mixing ratio of the total amount of $(1 + w)$ gm of moist air.

[2] Rossby, C.-G., *Mass. Inst. Tech. Met. Papers*, **1**, 3, 1932.

$$\Theta_d = T\left(\frac{P}{p - e}\right)^{\frac{AR}{c_p{}^*}} \tag{13.6}$$

Upon comparing this equation with the definition for the potential temperature (9.1) and disregarding the slight difference between $c_p{}^*$ and c_p, it follows from (5.8) that

$$\Theta_d = \Theta(1 + 1.609w)^{\frac{AR}{c_p{}^*}} \tag{13.7}$$

14. Minimum Inversion. Because the density of water vapor is less than that of dry air, moist air is lighter than dry air at the same temperature and pressure. At a boundary surface where a drier air mass lies over a moister one a certain minimum inversion is therefore required in order to maintain the purely mechanical equilibrium, as shown by Margules.[1]

Let ρ be the density of the air below and $\rho + \Delta\rho$ the density of the air above the moisture discontinuity. The condition of mechanical equilibrium requires that the density decrease with the altitude. The stability condition is then expressed by

$$\Delta\rho \leqq 0$$

Because the pressure on both sides of the surface of discontinuity must be the same, the virtual temperature must increase,

$$\Delta T^* > 0$$

Upon substituting from (5.3), it follows that

$$\Delta T > -\frac{0.379}{p}\,T\,\frac{\Delta e}{1 - 0.379(e/p)} \tag{14.1}$$

If this condition were not fulfilled, the air above the discontinuity would be heavier than that below and the stratification would be unstable. When the water vapor decreases upward, as, for instance, on cloud surfaces, the temperature must increase. The inversion necessary to maintain stability in the case of a sudden decrease of the water-vapor content is called the "minimum inversion."

If the water vapor increases, the temperature may even decrease suddenly without offsetting the stable stratification. The actual temperature differences that result from (13.6) are,

[1] MARGULES, M., *Met. Z.*, Hann-vol., 243, 1906.

of course, quite small. With a pressure of 800 mb, a temperature of 280° abs, and a relative humidity of 100 per cent below and 50 per cent above the inversion, Δe is 5 mb because the maximum vapor pressure at 280° abs is 10 mb. Therefore, $\Delta T = 0.66°C$. The practical significance of such minimum inversions which ensure the stability of moisture discontinuities is therefore very slight.

Similar considerations apply also to the case where a gradual change of temperature and humidity in the vertical direction is taking place. It can be shown[1] that in an unsaturated moist air column the lapse rate of the virtual temperature, instead of the lapse rate of the temperature, must be lower than the adiabatic in order to have a stable stratification.

15. Variation of the Dew Point with the Altitude. Condensation Level. It was shown in Sec. 9 that air which ascends adiabatically is cooled, for it does work expanding against the pressure of the surrounding atmosphere. Therefore, when unsaturated moist air ascends sufficiently high, it may finally reach a temperature at which the water vapor contained in the air represents the saturation value. The height at which saturation is reached is called the "condensation level." Actually, the condensation does not necessarily begin as soon as saturation is reached; supersaturation may occur (see Sec. 16). Nevertheless, the condensation level and the corresponding condensation pressure give at least an idea as to the lowest possible height at which condensation and therefore cloud formation may be expected when air ascends adiabatically.

To obtain a working formula for the condensation level, it will be assumed that a quantum of air ascends adiabatically without mixing with its surroundings so that its potential temperature and its mixing ratio remain unchanged. Before the air is lifted, its dew point τ is lower than its temperature T, for the air is at first unsaturated. During the ascent, not only the temperature, but also the dew point decreases. The variation of the dew point with altitude may be obtained from (5.5) which gives the relation between maximum water-vapor pressure and temperature. In this formula the maximum vapor pressure e_m may be replaced by the actual vapor pressure e

[1] BRUNT, D., "Physical and Dynamical Meteorology," 2d ed., p. 44, Cambridge University Press, London, 1939.

and the temperature by the dew point τ so that, by logarithmic differentiation and using the constants for e_m over water,

$$\frac{de}{e} = \frac{7.5 \ln 10 \cdot 237.3\, d\tau}{(237.3 + \tau)^2}$$

Because the mixing ratio remains constant during an adiabatic process as long as the air is unsaturated it follows from (13.2) that

$$\frac{de}{e} = \frac{dp}{p}$$

Combining this with the hydrostatic equation[1]

$$\frac{dp}{p} = -\frac{g}{RT'}\,dz$$

the variation of the dew point in a rising air mass may be written

$$-\frac{d\tau}{dz} = \frac{g}{R}\,\frac{1}{7.5 \ln 10 \cdot 237.3}\,\frac{(237.3 + \tau)^2}{(273 + t)} \qquad (15.1)$$

when dew point and temperature are expressed in degrees centigrade. If second and higher powers of $\tau/237.3$ and $t/273$ are neglected, (15.1) becomes

$$-\frac{d\tau}{dz} = 1.71 \times 10^{-3}\left(1 + \frac{2\tau}{237.3} - \frac{t}{237}\right) \qquad (15.11)$$

Thus the dew point of an ascending quantum of air decreases much more slowly than its temperature.

At a height z (below the condensation level) the dew point is therefore approximately

$$\tau = \tau_0 + \left(\frac{d\tau}{dz}\right)_m z$$

where τ_0 is the dew point at the surface and $\left(\frac{d\tau}{dz}\right)_m$ the mean change of the dew point up to the level z. The variation of the temperature of the ascending air mass with height is approximately

$$t = t_0 - \Gamma z$$

where Γ is the adiabatic lapse rate. At the height of the condensation level h_c,

$$\tau_c = t_c$$

[1] T is here the temperature not of the ascending but of the surrounding air.

This gives for h_c

$$h_c = \frac{t_0 - \tau_0}{\Gamma + (d\tau/dz)_m} \tag{15.2}$$

or approximately

$$h_c = 121(t_0 - \tau_0) \quad \text{in meters} \tag{15.21}$$

In Sec. 18, it will be shown how the pressure and temperature at the condensation level may be determined graphically. Nevertheless the formula given here will be useful for estimates of the height at which condensation may occur.

The validity of the approximation formula (15.21) depends very much on the original assumption that no mixing between the ascending and the surrounding air takes place so that its humidity mixing ratio does not change.[1] This obviously can not be strictly true. In order to allow for changes of the humidity mixing ratio due to mixing with the surrounding air, it has been suggested[2] that not the mixing ratio at the surface but a mean value for the whole layer through which the air ascends should be considered, in order to allow for a decrease of the mixing ratio in the ascending air due to turbulent mass exchange with the environment.

16. The Role of the Condensation Nuclei. Atmospheric condensation takes place mostly in the form of water droplets; only near the ground, condensation on surfaces (plants, rocks, buildings, etc.) may sometimes occur. Owing to the surface tension the maximum vapor pressure over curved surfaces is larger than over plane water surfaces. Therefore, when saturation has just been attained with respect to a plane water surface the vapor pressure is still smaller than the saturation vapor pressure over a curved surface, and a small water droplet brought into such an atmosphere would evaporate. If e' is the water-vapor pressure over the curved surface, e that over the plane surface, μ the constant of capillarity, m_w the molecular weight of water vapor, σ the density of liquid water, T the temperature, and r the radius of the spherical droplet, it can be shown[3] that

[1] PETTERSSEN, S., *J. Aeronaut. Sci.*, **3**, 305, 1936; "Weather Analysis and Forecasting," p. 54, McGraw-Hill Book Company, Inc., New York, 1940.

[2] WOOD, F. B., *Mass. Inst. Tech.*, Met. course, Prof. notes, 10, 1937.

[3] PRESTON, T., "Theory of Heat," 3d ed., p. 392, Macmillan & Company, Ltd., London, 1919.

$$\frac{R^*}{m_w} T\sigma \ln \frac{e'}{e} = e' - e + \frac{2\mu}{r} \tag{16.1}$$

When e'/e is not very different from unity, (16.1) may be written in the simpler form which was originally derived by W. Thomson[1]

$$\frac{e' - e}{e} = \frac{2\mu m_w}{rR^*T\sigma} \tag{16.11}$$

The capillarity constant μ of water in contact with air depends on the temperature T as shown by the following table:[2]

T, deg abs	268	273	278	283
μ, dynes/cm	76.4	75.6	74.8	74.2

These values hold for a plane surface. The possible variation of μ due to the curvature of the liquid surface will be neglected. As long as the radius of the drop is large, e' is not very different from e. But it increases considerably when the size of the droplet decreases. The ratio $100 \times (e'/e)$ given in the following table represents the supersaturation with respect to a plane water surface when the water vapor is in equilibrium with respect to water droplets. The temperature is assumed to be 273° abs.

Radius of the drop, cm	10^{-2}	10^{-4}	10^{-5}	10^{-6}	5×10^{-7}	3×10^{-7}	10^{-7}
Supersaturation $100 \times (e'/e)$, per cent	100.001	100.12	101.2	112.7	127.0	148.9	330.4

This table shows that the atmospheric condensation would require enormous supersaturation if there were no other forces counteracting the effect of the surface tension. Actually, hygroscopic particles are always present in the atmosphere as has been shown by H. Koehler,[3] who found that salts are dissolved in the atmospheric condensation products. A large part of the nuclei, as, for instance, those consisting of sodium chloride, may originate by evaporation of ocean spray. Another impor-

[1] THOMSON, W., *Phil. Mag.*, **42** (4), 448, 1871.

[2] "Smithsonian Physical Tables," 8th ed., Table 193, the Smithsonian Institution, Washington, D. C., 1933.

[3] KOEHLER, H., *Medd. Stat. Met. Hydr. Anst.*, **2**, 5, 1925; *Geofys. Pub.*, **2**, 1, 6, 1921; *Gerl. Beitr. Geophys.*, **291**, 68, 1931. For a complete discussion of the present knowledge about nuclei see H. Landsberg, "Ergebn. d. kosm. Physik," Vol. 3, Akademische Verlagsgesellschaft, Leipzig, 1938; and M. G. Bennett, *Quart. J. Roy. Met. Soc.*, **60**, 3, 1934.

tant source of nuclei, especially those containing sulphuric products, is combustion due to industries. The normal sizes of the nuclei may lie between 7×10^{-7} and 10^{-5} cm. The number of nuclei varies, according to observations, between the orders of magnitude 10^0 and 10^6 per cubic centimeter. But only between 10^2 and 10^4 nuclei per cubic centimeter may be effective in the formation of clouds because the supersaturation in the atmosphere is much smaller than in the nuclei counters so that only on the largest and most hygroscopic nuclei are droplets formed.[1]

When condensation occurs on such hygroscopic particles, the droplets that are formed represent solutions of the hygroscopic substance in water. The water-vapor pressure e'' over a solution is lower than the pressure e over pure water. For a solution of low concentration containing m' molecules of the solute and m molecules of water, it can be shown that

$$e'' - e = -e \frac{m'}{m + m'} \tag{16.2}$$

Equation (16.2) shows that the equilibrium pressure becomes smaller the more concentrated the solution. For high concentrations, (16.2) no longer holds, but the saturation vapor pressure continues to decrease with increasing concentration.

Thus condensation nuclei can exist in the atmosphere in the form of watery solutions of high concentration, for some water vapor is always present in the atmosphere. When the atmospheric water-vapor pressure increases over the saturation value with respect to the watery nucleus, more condensation takes place; the droplet grows even if the relative humidity of the air is less than 100 per cent with respect to a plane water surface.

Small droplets containing a sufficient amount of the solute may therefore originate at relative humidities of less than 100 per cent. This has been demonstrated by Pick,[2] who showed that fogs frequently occur at relative humidities of less than 100 per cent. At first the droplets must remain very small, for only then can the depression of the vapor pressure due to the concentration of the solution balance the effect of the surface tension. For every nucleus, there exists a critical relative humidity of the air. When this value is reached, the drop can

[1] Findeisen, W., *Met. Z.*, **55**, 121, 1938.
[2] Pick, W. H., *Quart. J. Roy. Met. Soc.*, **57**, 238, 1926.

continue to grow without further increase of the relative humidity because the effects of the surface tension and of the solute on the saturation pressure balance each other. When the critical value of the relative humidity is surpassed, further condensation depends on the amount of water vapor released by the atmosphere when the air is cooled. The drops will finally grow by coagulation. Whether electric charges of the drops are important for coagulation, as suggested by Schmauss and Wigand,[1] appears doubtful. Coagulation occurs mainly between drops of different size, for such drops fall with different speed and move, therefore, relative to each other. If the cloud elements were all of the same size, only turbulent motion could be effective in bringing about coagulation,[2] but it appears that the magnitude of the drops in a cloud varies widely.[3]

The observations of rain from clouds, however, seem to indicate that coagulation cannot produce drops of the size measured in rain. Bergeron[4] has therefore suggested the hypothesis that rain is formed when ice crystals fall through clouds of small water droplets. Because the vapor pressure over ice is lower than over liquid water (see Table I, Appendix), evaporated water condenses on the ice crystals. The ice continues to fall and melts in falling, so that it arrives as rain at the earth's surface. This hypothesis is supported by investigations of Peppler[5] and Findeisen.[6] These authors found that rain, especially rain with large drops, originates as ice condensation.

Because the presence of the condensation nuclei lowers the freezing point of the water considerably, the formation of ice crystals cannot take place on condensation nuclei. Separate sublimation nuclei must be present in the atmosphere. The nature of these sublimation nuclei is not yet known. But it seems obvious that they must be solid particles. The order of magnitude of their linear dimension has been estimated to 10^{-6} cm by Findeisen.[6] Their number per unit volume appears

[1] Schmauss, A., and Wigand, A., "Die Atmosphäre als Kolloid," F. Vieweg & Sohn, Braunschweig, 1929.

[2] Arenberg, D., *Bull. Am. Met. Soc.*, **20**, 444, 1939.

[3] Findeisen, *loc. cit.* Houghton, H. G., Papers in Physical Oceanography and Meteorology, Mass. Inst. Tech. and Woods Hole Ocean. Inst., **6**, 4, 1938.

[4] Bergeron, T., *Un. int. géod. géophys., mét. assoc.*, 156, 1933.

[5] Peppler, W., *Beitr. Phys. Atm.*, **23**, 275, 1936.

[6] Findeisen, *loc. cit.*

to be considerably smaller than the number of condensation nuclei.

The structure of the snow particles depends largely on the state of the atmospheric layers through which they have passed. Findeisen[1] has discussed how far it may be possible to draw from the form of the snow conclusions concerning the structure of the atmosphere. In particular, when ice crystals fall through clouds of supercooled water droplets, they coagulate with these droplets and change gradually into soft hail. It is therefore possible to deduce from the appearance of such snow particles the presence of supercooled water in the atmosphere, which makes icing on aircraft likely.

17. Adiabatic Changes in the Saturated State. Owing to the presence of hygroscopic nuclei, condensation does not begin suddenly when the air has ascended to the "condensation level" and attained a relative humidity of 100 per cent with respect to a plane surface of pure water. The actual start of the condensation depends on the physical and chemical properties of the nuclei. In order to obtain droplets of appreciable size, however, a sufficient amount of water vapor must be released by the atmosphere. This is possible only when the relative humidity is close to 100 per cent. Moreover, when the drops reach an appreciable size, the surface tension is very small and the concentration of the solute very low, so that the influence of these two factors on the vapor pressure becomes negligible. For most purposes, condensation may therefore be assumed to begin at a relative humidity of 100 per cent.

The main effect of condensation on the adiabatic changes consists in a decrease of the rate of cooling of the air, for the cooling due to adiabatic expansion against the outer pressure is now partly compensated by the release of the latent heat of condensation.

For the theoretical discussion, it is necessary to assume either that the condensed water remains in the air and is carried along by the ascending air[2] or that all condensed water falls to the ground as precipitation.[3] When the condensed water falls

[1] Findeisen, W., *Met. Z.*, **56**, 429, 1939.

[2] Hertz, H., *Met. Z.*, **1**, 421, 1884.

[3] von Bezold, W., *Sitzb.-Ber. Akad. Wiss. Berlin*, p. 485, 1888. Neuhoff, O., *Abhandl. Preuss. Met. Inst.*, **1**, 6, 1901. Fjeldstad, J. E., *Geofys. Pub.*, **3**, 13, 1925.

out completely, the temperature changes that the air undergoes are evidently not reversible, for the air when descending again will be heated according to expression (13.5) for unsaturated moist air, *i.e.*, approximately at the dry-adiabatic rate. This process is called "*pseudo-adiabatic.*" On the other hand, when the condensed water is retained in the air, a fraction of the temperature increase in the descending air will be used to evaporate the liquid water again during the descent. In this case, which is called "saturated adiabatic" or "moist-adiabatic," the rate of the temperature increase during the descent is equal to the rate of the temperature decrease during the ascent. The process is reversible.

In the following derivation of the thermodynamic relations, which describe the change of moist air, it will be assumed that the condensation products are retained in the air so that the variations are saturated adiabatic.

It will first be assumed that all processes take place at temperatures above the freezing point. This is frequently referred to as the *rain stage.*

According to the assumption, the total water content $\bar{w}$ per gram of air is constant. It consists of an amount of water vapor w and of liquid water w_l;

$$\bar{w} = w + w_l$$

From the first law of thermodynamics for dry air, it follows that

$$dq_d = c_p\, dT - ART \frac{d(p - e)}{p - e}$$

Let the heat added to the total water content $\bar{w}$ be dq_w. This heat is used in three ways.

1. The amount $c(\bar{w} - w)\, dT$ increases the temperature of the liquid water. (The specific heat of the water c may be regarded, with sufficient accuracy, as constant and equal to unity.)

2. The amount $L\, dw$ is used for the evaporation of liquid water ($L = 595 - 0.5t$°C, the heat of condensation).

3. The amount $c_s w\, dT$ increases the temperature of the saturated water vapor (c_s is the specific heat of saturated water vapor).

Thus, if the change of the volume of the liquid water is neglected,

$$dq_w = c(\bar{w} - w)\, dT + L\, dw + c_s w\, dT$$

To find the specific heat of saturated water vapor c_s, the fact may be used that the entropy is a total differential, as shown in Sec. 22. The entropy change of the water vapor and liquid is dq_w/T. This expression is a total differential if

$$\frac{\partial}{\partial w} \frac{c(\bar{w} - w) + c_s w}{T} = \frac{\partial}{\partial T} \left(\frac{L}{T} \right)$$

Therefore,

$$c_s = c + T \frac{d}{dT} \left(\frac{L}{T} \right) \tag{17.1}$$

which gives the specific heat of saturated water vapor. Thus

$$dq_w = c\bar{w}\, dT + Td \left(\frac{Lw}{T} \right)$$

Upon adding the expression for the change of heat of the dry air and of the water vapor, it is found that the total change of the heat content of the moist air

$$dq = (c_p + c\bar{w})\, dT + Td \left(\frac{Lw}{T} \right) - ART \frac{d(p - e)}{p - e} \tag{17.2}$$

For adiabatic changes, *i.e.*, when $dq = 0$, this equation may be integrated and becomes—when the index 0 indicates the initial stage—

$$\ln \frac{p - e}{p_0 - e_0} = m' \ln \frac{T}{T_0} + \frac{1}{AR} \left(\frac{Lw}{T} - \frac{L_0 w_0}{T_0} \right) \tag{17.3}$$

Here

$$m' = \frac{c_p + c\bar{w}}{AR} \approx 3.47(1 + 4.18\bar{w}) \tag{17.4}$$

Equation (17.3) may be written in different forms. If the pressure is eliminated by

$$w = 0.621 \frac{e}{p - e} \tag{5.8}$$

it is transformed into

$$\ln \frac{w}{w_0} - \ln \frac{e}{e_0} + m' \ln \frac{T}{T_0} + \frac{1}{AR} \left(\frac{Lw}{T} - \frac{L_0 w_0}{T_0} \right) = 0 \tag{17.5}$$

and if w is eliminated, it changes into

$$\ln \frac{p - e}{p_0 - e_0} - \frac{0.621}{AR} \left[\frac{Le}{T(p - e)} - \frac{L_0 e_0}{T_0(p_0 - e_0)} \right] - m' \ln \frac{T}{T_0} = 0 \tag{17.6}$$

It should be noted that in Eqs. (17.3), (17.5), and (17.6) w is a function of pressure and temperature only, for saturation prevails so that e is the saturation vapor pressure and therefore a function of the temperature only.

When the adiabatic, or, to be more accurate, the moist-adiabatic, ascent of the air continues sufficiently long, the air will eventually reach the temperature of 0°C. It is known today that liquid water exists quite frequently in the supercooled state in the atmosphere. According to Findeisen,[1] water clouds are more frequent than ice clouds down to a few degrees below freezing. Below about −10°C, ice clouds become more numerous. Water droplets have been observed down to temperatures as low as −40°C.

Before it was known that supercooling is a rather regular occurrence in the atmosphere, the treatment of adiabatically ascending air reaching the freezing point was based on the assumption that all liquid water begins to freeze and that the temperature remains constant when the air ascends farther owing to the released heat of fusion until all the liquid water has been changed over in the solid state. The name *hail stage* was given to this form of the adiabatic changes. In practice the hail stage is today mostly disregarded, for the phenomenon of supercooling shows that it frequently does not take place. Moreover, when the changes during the ascent are pseudo-adiabatic, *i.e*, when all the condensed water falls out, the ascending air can obviously not pass through the hail stage.

Below the freezing point the cooling of the moist air produces snow crystals during the adiabatic ascent if supercooling does not occur. Therefore, the stage is called the *snow stage.*

The equations for the snow stage are analogous to the equations for the rain stage except that, in all considerations of the rain stage, ice now takes over the role of the liquid water. Consequently the constants for the water have now to be replaced by the corresponding constants for ice. Because the specific heat of ice $c_i = 0.49$ at 0°C, it follows that the constant

$$m'' = \frac{c_p + c_i\bar{w}}{AR} \approx 3.47(1 + 2.05\bar{w}) \tag{17.7}$$

will replace m', which appeared in Eqs. (17.3), (17.5), and (17.6).

[1] FINDEISEN, W., *Met. Z.*, **55**, 121, 1938.

Instead of the heat of condensation L the heat of sublimation L_s appears. According to Fjeldstad,[1]

$$L_s = 677 \text{ cal/gm}$$

and this value may be regarded as constant. Thus when the water-vapor content of the air is expressed by the vapor pressure the equation for adiabatic changes in the snow stage is given by the equation

$$\ln\left(\frac{p-e}{p_0-e_0}\right) - \frac{0.621}{AR}\left(\frac{L_s e}{T}\frac{1}{p-e} - \frac{L_s e_0}{T_0}\frac{1}{p_0-e_0}\right) - m'' \ln\frac{T}{T_0} = 0 \tag{17.71}$$

or when the water-vapor content is expressed by the mixing ratio w,

$$\ln\frac{p-e}{p_0-e_0} = m'' \ln\frac{T}{T_0} + \frac{L_s}{AR}\left(\frac{w}{T} - \frac{w_0}{T_0}\right) \tag{17.72}$$

18. The Application of the Equations for Saturated Adiabatic Changes to Atmospheric Processes. Pseudo-adiabatic Chart. In applying the results of Sec. 17 to the atmosphere, it is necessary to keep in mind the assumptions on which the derivation of the equations was based. The assumption that the changes are adiabatic, and its limitations, were discussed in Sec. 8.

It was furthermore assumed that all the water vapor which is transformed into the liquid or the solid state is retained in the air. This is obviously not always the case, for precipitation does occur in the atmosphere. The assumption that all condensation products are retained is not any more justified than the other extreme that they are expelled completely, *i.e.*, that the changes are pseudo-adiabatic. The first alternative was preferred for reasons of mathematical expediency. When *all* the water is retained, the total *constant* amount of water $\bar{w}$ appears in (17.2) whereas when the changes are supposed to be pseudo-adiabatic $\bar{w}$ has to be replaced by w, the *variable* mixing ratio of the water vapor. In practice, the difference between the pseudo-adiabatic and the saturated-adiabatic ascent is, however, negligible at temperatures above freezing; for $\bar{w}$ as well as w is small. Only when the freezing temperature is reached does the difference become noticeable, for the air does not go through the hail stage

[1] FJELDSTAD, J. E., *Geofys. Pub.*, **3**, 11, 1925.

when its changes are pseudo-adiabatic but passes directly from the rain stage to the snow stage. Furthermore, when the ascent is pseudo-adiabatic, any subsequent descent will follow the dry adiabatic for all the condensed water is being expelled during the ascent so that the air becomes unsaturated as soon as it begins to descend.

The actual computation of the simultaneous changes of pressure, temperature, and moisture content are greatly facilitated by the use of thermodynamic charts. A simple thermodynamic chart which permits one at least to find the temperature variations of dry or unsaturated moist air undergoing adiabatic pressure changes has already been described in Sec. 9. The abscissa of this chart is the temperature and the ordinate the logarithm of the pressure or its 0.288th power.[1]

With the aid of the relations developed in Sec. 17, lines can be constructed that give the simultaneous values of pressure and temperature of ascending saturated air. These lines are called "saturated adiabats" or "pseudo-adiabats"; for the difference between the two types of changes is negligible, as pointed out previously. The resulting chart may be referred to as a "saturated-adiabatic" or a "pseudo-adiabatic" chart.

The first saturated-adiabatic chart has been constructed by Hertz.[2] It also takes into account the hail stage whose representation has been omitted on the charts now in use for the reasons stated above (page 48). Because the hail stage is omitted on this chart, the designation "pseudo-adiabatic" is preferable. Figure 8 shows a pseudo-adiabatic chart. The abscissas are the temperatures in degrees centigrade and the ordinates the 0.288th powers of the pressures in millibars. The sloping straight lines are the dry adiabats computed in the manner explained in Sec. 9.

For the construction of the saturated adiabats, (17.6) can be used at temperatures above the freezing point. For each moist adiabat the initial value of p and T may be taken arbitrarily. The saturation pressure e and the heat of condensation L are determined by T. Therefore, one value of T belongs to every given value of p. A mean value may be assumed for $\bar{w}$, for

[1] Strictly speaking the chart with the coordinate $p^{0.288}$ should not be called a thermodynamic chart (see p. 81).

[2] HERTZ, *loc. cit.*

it has little effect on the size of m owing to its smallness. In view of the slight difference between saturated- and pseudo-adiabatic changes the curves constructed with the aid of (17.6) may also be regarded as pseudo-adiabats. Below the freezing point the computation of the saturated adiabats may be continued with the aid either of the same equation (17.6), when it is assumed that supercooling takes place, or of equation (17.9). The practical difference arising from the use of these two assumptions is negligible.

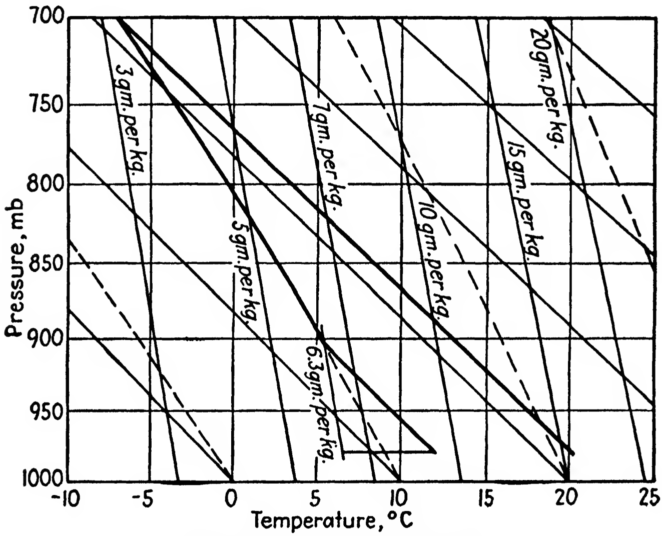

FIG. 8.—Pseudo-adiabatic chart.

The saturated adiabats computed in this manner are drawn as broken curves in Fig. 8. They are sloping more steeply than the dry adiabats, a fact indicating that the adiabatic temperature changes due to pressure changes are smaller in the saturated than in the unsaturated stage. This condition is, of course, caused by the heat of condensation which compensates partly for the temperature change due to the adiabatic expansion or contraction. The compensation is larger the greater the available amount of heat of condensation. Consequently, the saturated adiabats are steeper for high temperatures and pressures where the saturation mixing ratio is larger than for lower temperatures. For very low temperatures where the mixing

ratio is practically negligible the saturated adiabats are practically parallel to the dry adiabats.

For many computations, it is desirable to have on such a chart, also, lines representing the atmospheric moisture content for a given pressure and temperature when saturation prevails. On the chart under discussion the lines of constant saturation mixing ratio have been plotted for this purpose (full curves; steeper than the dry adiabats). These lines are constructed by means of the equation

$$w = 0.621 \frac{e}{p - e} \tag{5.8}$$

Because a state of saturation is assumed, e is the saturation pressure of water vapor which depends on the temperature only, so that w is only a function of p and T. Below the freezing point, the saturation vapor pressure either over water or over ice may be chosen; for, quite frequently, water occurs in supercooled form in the atmosphere, also.[1] The choice is not important; for at first the difference between the two vapor pressures is quite small, and at lower temperatures w itself becomes insignificant. In Fig. 8 the saturation pressure over water has been used. The figures give the mixing ratio in grams of water vapor per kilogram of dry air. For a temperature of 12°C and a pressure of 980 mb, for example, the saturation mixing ratio is 9 gm of water vapor per kilogram of dry air. When the relative humidity is not 100 per cent, the mixing ratio is obtained by multiplication of the saturation mixing ratio with the relative humidity. This, though not strictly correct, is sufficiently accurate; for $e \ll p$. Thus the mixing ratio of air whose pressure, temperature, and relative humidity are given can easily be obtained from the chart.

The numerous practical uses of the pseudo-adiabatic charts can be explained best by an example. Consider a unit of air at a pressure of 980 mb, a temperature of 12°C, and a relative humidity of 70 per cent. The saturation mixing ratio of this air is 9 gm/kg, and its actual mixing ratio 6.3 gm/kg. Its dew point τ is the temperature to which it would have to be cooled at constant pressure so that its mixing ratio 6.3 gm/kg becomes the saturation mixing ratio. Upon following the isobar 980 mb to

[1] The effects of surface tension and impurities of the water on the saturation pressure are not taken into account.

the saturation mixing ratio 6.3 gm/kg, it is found that τ = 6.6°C. Next, the pressure and temperature at the condensation level may be determined where the air becomes saturated. Because the air is in the unsaturated stage, it follows the dry adiabat through the starting point and its mixing ratio remains constant up to the condensation level, according to Sec. 13. Saturation occurs when the mixing ratio becomes the saturation mixing ratio owing to the adiabatic temperature decrease. Consequently, the condensation point is situated at the intersection of the dry adiabat through the initial point with the saturation mixing-ratio line representing the actual mixing ratio of the air. In the example its pressure is 904 mb, and its temperature 5.6°C. These are the condensation pressure and temperature. It may be noted that the dew point remains always on the same saturation mixing-ratio line during the unsaturated-adiabatic ascent. Therefore, these lines are also called "dew-point lines." When the air continues its adiabatic ascent, it will now follow the saturated adiabat through the condensation point. The temperature 0°C is reached at a pressure of 805 mb. Completely dry air of the same initial pressure and temperature would have been cooled to the freezing point already at a pressure of 842 mb. Suppose the air ascends to a height where the pressure is 700 mb. Its temperature would here be −7°C. The saturation mixing ratio is here 3.2 gm/kg of dry air as compared with the original 6.3 gm/kg. The surplus of 3.1 gm/kg either is present as liquid water and ice (snow) in the atmosphere or has fallen out in part or completely as precipitation. In this manner the chart affords also a rough estimate of the possible precipitation intensity.

When all the liquid water and ice (snow) is retained, the air will undergo the same states in reverse order when it is brought down to its original pressure. When, on the other hand, all liquid and ice are precipitated, *i.e.*, when the process is pseudo-adiabatic, the air becomes immediately unsaturated when descending and follows the dry adiabat through the point at which the descent started. In the example chosen here, the air would attain a temperature of 20.1°C if brought back to its original pressure of 980 mb provided that it has discharged all its condensation and sublimation products. Its relative humidity is now only about 21 per cent. The temperature of the air has risen during the ascent and subsequent descent from 12° to

20.1°C. This temperature rise is due to the latent heat of condensation that has been released during the ascent. It may also happen that the air loses part of its water and snow content during the ascent so that the descent is first saturated adiabatic down to a certain pressure and then dry adiabatic.

The best known examples for pseudo-adiabatic changes are furnished by the foehn and Chinook winds of mountainous regions, when the air ascends over a mountain range and descends on the other side.[1] During the ascent the air cools at the saturated-adiabatic rate and loses at least a great part of its water or snow so that on its descent it follows a dry adiabat during the whole or the latter part of the descent. The foehn appears therefore as a warm dry wind.

19. Saturated-adiabatic Lapse Rate. The differential equation for dry-adiabatic changes (8.4) led to the expression (9.2) for the rate of the temperature change with the altitude of ascending or descending dry air and to the concept of the dry-adiabatic lapse rate. In a similar manner the equations of Sec. 17 may be used to find the rate of the temperature change with altitude in ascending saturated air which leads to the saturated-adiabatic lapse rate.

The expression for the saturated-adiabatic lapse rate can, however, be derived more directly, although the derivation is not quite rigorous. According to (8.2),

$$dq = c_p\,dT - \frac{ART}{p}\,dp$$

for dry or unsaturated moist air. When condensation begins, let the decrease of the mixing ratio be dw, so that the heat $-L\,dw$ is released and increases the temperature of the air. The minus sign indicates that heat is released when the mixing ratio decreases. When the presence of water vapor and liquid water is neglected (except for its production of the latent heat of condensation), it follows that

$$-L\,dw = c_p\,dT - \frac{ART}{p}\,dp \tag{19.1}$$

Equation (19.1) may be compared with (17.2) to ascertain the simplifications involved. These simplifications are permissible

[1] Lammert, L., *Arb. Geophys. Inst. Leipzig*, **2**(7), 261, 1920. Krick, I. P., *Gerl. Beitr. Geophys.*, **39**, 399, 1933.

here; for only small changes of p, T, and w are considered, whereas in Sec. 17 an integration was to be performed.

If the pressure variations are due to vertical motions,

$$\frac{dp}{\rho} = -g\,dz \tag{6.1}$$

According to (5.82),

$$\frac{dw}{w} = \frac{de}{e} - \frac{dp}{p} \tag{5.82}$$

Upon substituting in (19.1) and rearranging, it follows that

$$\Gamma' = -\frac{dT}{dz} = g\,\frac{A + 0.621\,\dfrac{e}{p}\dfrac{L}{RT}}{c_p + 0.621\,\dfrac{L}{p}\dfrac{de}{dT}} \tag{19.2}$$

In (6.1) the density of the environment appears. In substituting (6.1) in (19.1), it has been assumed that the temperature of the ascending air is not very different from the temperature of the environment so that both temperatures may be regarded as equal.

If the snow stage is considered, L in (19.2) stands for the heat of sublimation.

The expression Γ' is called the "saturated-adiabatic" or "moist-adiabatic lapse rate." Its numerical value depends on the temperature and pressure. The term de/dT may be obtained by differentiating (5.5) or using Table I of the Appendix for the saturation vapor pressure. The following table shows the saturated-adiabatic lapse rate in degrees centigrade per 100 m for a few temperatures and pressures:

T, deg abs	p = 500 mb, deg C/100 m	p = 1000 mb, deg C/100 m
243	0.88	
263	0.61	0.74
283	..	0.51
303	..	0.36

The values of the lapse rate below freezing have been computed for saturation with respect to ice. A graph showing the satu-

rated-adiabatic lapse rates for different pressures and temperatures has been given by Brunt.[1]

20. Stability with Respect to Saturated-adiabatic Changes. Conditional Instability. The saturated-adiabatic lapse rate plays a similar role in the concept of stability with reference to saturated-adiabatic changes as does the dry-adiabatic lapse rate in the case of dry-adiabatic changes. In particular, a column of air with a lapse rate α is in stable or unstable equilibrium with respect to saturated-adiabatic changes when $\alpha < \Gamma'$ or $\alpha > \Gamma'$, respectively. There is, however, a complication owing to the moisture content of the environment. When saturated air of the initial temperature T is lifted from z to $z + \Delta z$, its temperature becomes[2]

$$T - \Gamma' \Delta z$$

The temperature of the surrounding air at the level z may be T, the same as the temperature of the displaced air. The temperature of the surrounding air at $z + \Delta z$ is

$$T - \alpha \Delta z$$

where α is the lapse rate of the surrounding air. The lifted particle will be stable if its density is larger than that of the air surrounding it. The density depends not only on the temperature but also on the humidity. When the surrounding air is also saturated, stability exists if

$$\alpha < \Gamma'$$

and instability if

$$\alpha > \Gamma'$$

similar to the case of dry air considered in Sec. 9. If the surrounding air is not saturated, the lifted particle of air is not necessarily heavier, even though α may be smaller than Γ' and its temperature therefore smaller than the temperature of the surrounding air, for the moister air is lighter than the drier air at the same pressure and temperature. It is evidently not sufficient that $\alpha < \Gamma'$; α must be smaller by a finite amount that

[1] Brunt, D., *Quart. J. Roy. Met. Soc.*, **59**, 351, 1933; "Physical and Dynamical Meteorology," p. 66.

[2] The temperature variation of the ascending air depends slightly on the temperature of the surrounding air, also, as was seen in Sec. 8. But this small effect may be neglected here.

depends on the difference in moisture content between the air and its environment. Thus, although the saturated air will be unstable when $\alpha > \Gamma'$, it is not necessarily stable when $\alpha < \Gamma'$. However, in the latter case it will at least not be very far from stability. Therefore, the saturated-adiabatic lapse rate can in practice be regarded with sufficient accuracy as the limit between stable and unstable stratification.

When the temperature lapse rate of an air column is intermediate between the dry- and the moist-adiabatic unsaturated air lifted adiabatically would always be cooler than its surroundings and would therefore be in a stable position. Saturated air would be warmer and, therefore, in an unstable position. This type of equilibrium which is very common in the atmosphere is referred to as "conditional instability."

CHAPTER IV

FURTHER APPLICATIONS OF THERMODYNAMICS TO THE ATMOSPHERE

21. The Energy of Thermodynamic Processes. The Carnot Cycle. The atmosphere as a whole may be compared with a heat engine that is maintained in motion by being heated in the tropics and cooled in the polar zones. Similarly, the thermodynamical processes occurring in many of the smaller scale atmospheric circulations, as, for instance, monsoons or land and sea breezes, may be treated as the actions of a heat engine in which air is the working substance.

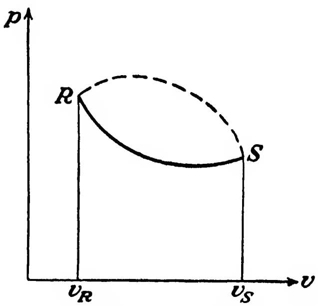

FIG. 9.—Indicator diagram (pv diagram).

The work done by a gas serving as the working substance in a heat engine can be studied by means of a so-called "indicator" diagram. The coordinates of such a diagram are the specific volume and the pressure (Fig. 9). The full curve RS in Fig. 9 may represent the successive states of the gas while it is changing from the state represented by R to S. Because the gas expands in the direction from R to S, it does work against the outer pressure which will be counted negative; work done *on* the gas will be counted positive. The amount of work done by the gas while it expands from R to S is given by (*cf.* page 18)

$$-\int_{v_R}^{v_S} p\, dv \tag{21.1}$$

It will be noted that the amount of work depends not only on the initial and the final state but also on the path itself. If the gas is then brought back to its original state R along the dotted line RS, the amount of work done by the gas is

$$-\int_{v_S}^{v_R} p\, dv \tag{21.11}$$

The whole process that the gas undergoes is called a cyclic process. The total work done by the gas during the cyclic process is

$$-\int_{v_R}^{v_S} p\,dv - \int_{v_S}^{v_R} p\,dv = -\oint p\,dv \tag{21.12}$$

The circle through the integral on the right-hand side of this equation indicates that the integration is to be performed along a closed curve. When the cycle represented by this curve is completed, the gas has returned to its original state. The total amount of work done is measured by the area bounded by the full and broken curves RS, for the area enclosed between the full curve RS and the abscissa appears in the first integral on the left-hand side of (21.12) with a negative sign and in the second integral with a positive sign. If the cyclic process is performed in the direction assumed here, counterclockwise, the total work done is positive (*i.e.*, the work done *by* the gas is negative), for the area under the full curve is smaller than the area under the broken curve. The work done on the gas is larger than the work done by the gas.

To compute the total work, or energy expended, during a variation of the gas the thermodynamical law that is followed during the changes of state has to be specified. One of the most important cyclic processes in thermodynamics is the Carnot cycle, which consists of two isotherms and two adiabats. The heat engine performing such a Carnot cycle may have two heat reservoirs of the temperatures T' and T'', which are so large that their temperatures are not changed when heat is added to or taken away from them in order to cool or warm the working substance. The working substance is an ideal gas. It may be contained in a cylinder one end of which is closed by a piston. This piston can move without resistance so that no energy is lost by friction. The changes of state of the working substance are reversible so that every change, for instance, expansion due to reduction of the outer pressure acting on the piston, takes place so slowly that the piston does not gain any kinetic energy.

The cyclic process that the gas undergoes is represented in Fig. 10. The gas may at first be in the state characterized by the point 1 where its temperature is T'. The temperature of the gas is kept constant at the temperature T' by a suitable connection with the reservoir of the temperature T'. By reduction

of the outer pressure, it is made to expand. Thus, the gas does work against the outer pressure. The heat equivalent of the work is furnished by the reservoir of the temperature T'. Because the change of state is isothermal, the work done by the gas when it expands from 1 to 2 given by the expression

$$-W_{12} = A \int_1^2 p\,dv = ART' \ln \frac{v_2}{v_1} \tag{21.2}$$

Because the heat q_{12} added to keep the gas at constant temperature is equal to the work done by the gas, it follows further that[1]

$$-W_{12} = q_{12} \tag{21.21}$$

Next the gas may expand adiabatically until it reaches the

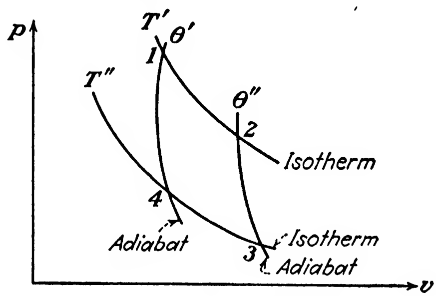

FIG. 10.—Carnot cycle.

temperature of the second reservoir T'' ($T'' < T'$). Its representation on the indicator diagram describes thereby an adiabat from the point 2 to 3. Consequently

$$q_{23} = 0 \tag{21.3}$$

and, according to the first law of thermodynamics for ideal gases (8.1),

$$-W_{23} = A \int_2^3 p\,dv = c_v(T' - T'') \tag{21.31}$$

Now, the gas is connected with the second reservoir T'' and is compressed isothermally to the specific volume v_4 from which the original specific volume v_1 can be reached by adiabatic compression. Here

$$W_{34} = -q_{34} \tag{21.4}$$

[1] The heat as well as the work added to a system will be counted positive.

The amount of heat q_{34} is taken away from the gas and added to the reservoir of the temperature T''.

Further,

$$W_{34} = -A \int_3^4 p\,dv = -ART'' \ln \frac{v_4}{v_3} \tag{21.41}$$

Finally, the gas is compressed adiabatically from the volume v_4 to the volume v_1, whereby

$$q_{41} = 0 \tag{21.5}$$

and

$$W_{41} = -c_v(T'' - T') \tag{21.51}$$

The substance has now been brought back to its original state. The total amount of work done is obtained by adding (21.2), (21.31), (21.41), and (2.51).

$$W = -ART \ln \frac{v_2}{v_1} - ART'' \ln \frac{v_4}{v_3}$$

for the contributions along the adiabats cancel each other. By virtue of the adiabatic relation (8.43),

$$\frac{T'}{T''} = \left(\frac{v_3}{v_2}\right)^{\lambda-1} = \left(\frac{v_4}{v_1}\right)^{\lambda-1}$$

Hence,

$$W = -AR(T' - T'') \ln \frac{v_2}{v_1} \tag{21.6}$$

Because $T' > T''$ and $v_2 > v_1$, $W < 0$; *i.e.*, work is done by the substance. In the indicator diagram the work is represented by the area 1234. While the working substance remains unchanged, a certain amount of heat q_{12} has been taken away from the first reservoir and the amount q_{34} has been added to the second reservoir. From (21.2) and (21.41), it follows that the absolute value of the heat taken away from the first reservoir is larger than the amount added to the second, for the temperature during the first isothermal change is higher than during the second.

The ratio of the work done by the gas to the amount of heat deducted from the first reservoir is called the "efficiency" η of the engine.

$$\eta = \frac{-W}{q_{12}} = 1 - \frac{T''}{T'} \tag{21.7}$$

22. Entropy. From the discussion of the Carnot cycle in the preceding section, it is seen that

$$\frac{q_{12}}{T'} + \frac{q_{34}}{T''} = 0 \tag{22.1}$$

A similar relation holds for any reversible cyclic process. In Fig. 11, such a process is represented in a p-v plane. This cycle may be broken up into a number of Carnot cycles, as indicated in the figure where the full curves represent the adiabats and the broken curves the isotherms. Along the adiabats, no heat is taken away from or added to the substance. Therefore, all that needs to be shown is that when the subdivision is made sufficiently small the amount of heat added or subtracted along the isotherm between two adiabats is equal to the amount of heat added or subtracted along the part of the original cycle between these two adiabats. Consider, for instance, the isotherm BC and the part AC of the original cycle. If ABC is regarded as a cyclic process, the total work performed during this cycle is represented by the area ABC and the heat added along AC and BC is proportional to the length of these curves. Along the adiabat AB, no heat is added. Therefore, the sum of the heat added along BC and along AC must be equal to the work done during the cycle change. This work is equal to the area of the triangle which becomes small of higher order than the length of the sides of the triangle when the adiabats and the isotherms are drawn at smaller and smaller intervals. It follows that

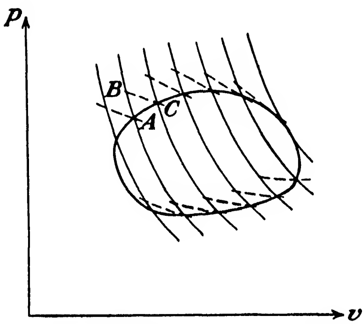

FIG. 11.—Entropy during a reversible cyclic process.

$$q_{BC} + q_{CA} = \text{quantity of higher order}$$

Consequently, when both curves are traversed in the same direction,

$$q_{BC} = q_{AC} + \text{quantity of higher order}$$

Thus, the heat added on the infinitely small line AC of the original cycle may be replaced by the heat added on the isotherm BC.

By repeating this reasoning for the whole cyclic process, it is seen that the result expressed by (21.1) for the Carnot cycle can be extended to an arbitrary reversible cycle.[1]
Hence,

$$\sum \frac{\Delta q}{T} = \text{quantity of higher order}$$

T is the temperature of the isotherm along which the small amount of heat Δq has been added to the system. When the steps between the points are made infinitely small, the summation has to be replaced by an integration, and the difference between the heat added along the cycle and along the isotherms vanishes.

$$\oint \frac{dq}{T} = 0 \tag{22.2}$$

From (22.2), it follows that $\int dq/T$ between two given points A_1 and A_2 is independent of the path. To show this, two such paths may be combined in a cyclic process by changing the direction in which the changes of state take place along one of the paths. When irreversible changes of state take place,

$$-\oint \frac{dq}{T} > 0$$

as shown in textbooks on thermodynamics.

The quantity $\int dq/T$ counted from an arbitrary origin is called the "entropy" of the system

$$\phi = \int \frac{dq}{T} + \text{const} \tag{22.3}$$

According to Nernst's theorem the entropy vanishes for $T = 0$, but in meteorology it is not necessary to take this fact into account. The importance of the entropy is due to the fact that it is independent of the path, *i.e.*, of the particular law which the change of state follows. From (22.2) it follows that $d\phi$ is a total differential. In Sec. 17, this fact was used to formulate the adiabatic relation for the ascent of saturated air.

[1] In a similar manner, it can be shown that the expression (21.6) for the efficiency of a heat engine performing a Carnot cycle holds for any reversible cyclic process.

Changes of state are called "isentropic" if $d\phi = 0$. They satisfy the condition that $dq = 0$ and are therefore always adiabatic. The converse is not necessarily true, for isentropic changes must also be reversible. The reversibility implies that the variation of state is so slow that it may be considered as a succession of equilibrium states. The so-called "adiabatic changes" in meteorology are really isentropic changes, for they are assumed to be reversible, as already pointed out in Sec. 8, where the equation for dry-adiabatic changes was derived.

If the first law of thermodynamics in the form (8.2) is substituted in (22.3), integration shows that the entropy of dry air

$$\phi - \phi_0 = c_p \ln \frac{T}{T_0} - AR \ln \frac{p}{p_0} \tag{22.4}$$

Here ϕ_0 is the arbitrary value of the entropy at the pressure p_0 and T_0. Shaw[1] has suggested that for meteorological purposes

$$\phi = 0$$

when $T = 100°$ abs. and $p = 1000$ mb would be a suitable choice.

When the definition of the potential temperature (9.1) is introduced, (22.4) may be written

$$\phi - \phi_0 = c_p \ln \frac{\Theta}{\Theta_0} \tag{22.5}$$

Thus the entropy of dry air is proportional to its potential temperature. Because c_p has the dimensions calorie per gram and degree, Eq. (22.5) gives the entropy in caloric units. To obtain it in mechanical units, it must be multiplied by the mechanical equivalent of heat.

The entropy of moist air can be obtained directly from (17.2). Division of this equation by T and integration show that

$$(\phi - \phi_0) = \int \frac{dq}{T} = (c_p + c\bar{w}) \ln \frac{T}{T_0} + \frac{Lw}{T} - \frac{L_0 w_0}{T_0} - AR \ln \frac{p - e}{p_0 - e_0} \tag{22.6}$$

The moist air may be regarded as consisting of dry air and water vapor. Hence, the entropy variation of the dry air

[1] Shaw, N., "Manual of Meteorology," Vol. 3, p. 247, University Press, Cambridge, 1933

$$(\phi - \phi_0)_{\text{dry air}} = (\phi - \phi_0)_{\text{moist air}} - c\bar{w} \ln \frac{T}{T_0} - \left(\frac{Lw}{T} - \frac{L_0 w_0}{T_0}\right) \tag{22.7}$$

When the moist air undergoes an isentropic process, it follows, because the second term on the right-hand side is small, that

$$(\phi - \phi_0)_{\text{dry air}} = -\left(\frac{Lw}{T} - \frac{L_0 w_0}{T_0}\right) \tag{22.71}$$

23. Energy Released by the Adiabatic Ascent of Air. When a parcel of air ascends or descends, its temperature and density are in general different from that of the surrounding air whereas the pressures are the same,

$$p' = p$$

The variables marked with a prime refer to the ascending quantum of air, the variables without a prime to the surrounding air. For the surrounding air the hydrostatic equation holds so that

$$0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{6.1}$$

The air in vertical motion is not in equilibrium with the air surrounding it, for its density ρ' is different from the density ρ of the environment. The acceleration to which the unit mass of the air in vertical motion is subjected is given by

$$a = -g - \frac{1}{\rho'}\frac{\partial p}{\partial z}$$

Upon substituting for $\partial p/\partial z$ from (6.1) the acceleration becomes

$$g\,\frac{\rho - \rho'}{\rho'} = g\,\frac{T' - T}{T} \tag{23.1}$$

When the air is moist, T' and T are virtual temperatures; but here the effect of the moisture content on the density may be neglected. The work done by the unit mass[1] when moved a distance dz

$$d\bar{W} = g\,\frac{T' - T}{T}\,dz \tag{23.2}$$

[1] It should be noted that we are considering now the work done by the ascending mass as positive, not the work done on the mass as in Sec. 21. To indicate this difference the notation $\bar{W}$ is used.

or, according to (6.11),

$$d\bar{W} = -AR(T' - T)\, d \ln p \tag{23.21}$$

The conversion factor A is added so that $\bar{W}$ is expressed in thermal units.

When the air ascends through a colder environment, $T' > T$; $d\bar{W} > 0$ during ascending motion, energy is liberated, whereas energy is expended when the air ascends through a warmer environment where $T' < T$.

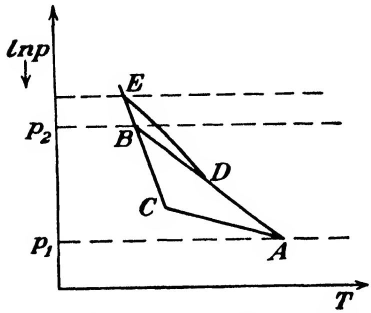

FIG. 12.—Energy released during the adiabatic ascent of dry or unsaturated air and of saturated air.

Equation (23.21) shows how the energy necessary to lift a parcel of air from a pressure level p_1 to another level p_2 may be represented on an adiabatic chart (T-ln p chart). In Fig. 12, ACB represents the observed temperature distribution. When air rises from p_1 to p_2, it follows the adiabat from A to B. At B, it is again in equilibrium. The work done by lifting the air is

$$-AR \int_{p_1}^{p_2} (T' - T)\, d \ln p$$

This integral is equal to the area enclosed by the adiabat AB and the temperature-pressure curve ACB. In the present example the temperature T' of the moving parcel of air is larger than the temperature T of the environment except at A and B while $d \ln p$ is negative. The integral is consequently positive; energy has been released during the ascent.

The preceding considerations may be extended to the vertical motions of moist saturated air. Equation (23.21) would remain unchanged, but the temperature T' of the parcel of ascending air would follow a moist adiabat instead of a dry adiabat as before. In Fig. 12 an example is also shown where the air first follows a dry adiabat from A to D. At D the saturation stage is reached and the air follows now the saturation adiabat DE. The external temperature distribution is indicated by $ACBE$. In this example the temperature is again always higher than that of the surround-

ing air so that energy is released. At E the temperature of the ascending air and that of the air surrounding it become equal again so that a new equilibrium position is reached.

The stability or instability of an atmospheric temperature distribution and the gain or loss of energy during vertical motion depend on the position of the pressure-temperature curve relative to the adiabat. When this curve is to the right of the adiabat, the atmospheric layer is in stable equilibrium and energy must be supplied to lift air, when the ascent curve is to the left of the adiabat, the equilibrium is unstable and energy is released during vertical motion.

24. Equivalent Potential Temperature and Equivalent Temperature. The adiabatic relation for dry air led to the definition of the potential temperature (Sec. 9), which remains constant as long as the changes of the air follow the dry adiabat. The usefulness of the potential temperature for air-mass analysis is, however, restricted, for the potential temperature varies when condensation takes place during the adiabatic process. But with the aid of the relations developed in Sec. 17, it is possible to introduce a quantity, the equivalent potential temperature, that remains constant during moist-adiabatic changes. The equivalent potential temperature is realized if the air is first lifted dry adiabatically to the condensation level, then lifted pseudo-adiabatically to the pressure zero so that the water vapor condenses and falls out, and is finally brought adiabatically to the standard pressure of 1000 mb. To formulate this definition of the equivalent potential temperature mathematically the slight difference between the rain and the snow stage can be disregarded. Furthermore, $c\bar{w}$ may be neglected with respect to c_p in Eq. (17.3). This does not cause a serious error and is correct in any case if the changes of state of the air are pseudo-adiabatic. It follows that

$$c_p \ln T + \frac{Lw}{T} - AR \ln (p - e) = C \tag{24.1}$$

where C is a constant. Upon introducing the partial potential temperature from (13.6) and putting $c_p^* \approx c_p$,

$$c_p \ln \Theta_d - AR \ln P + \frac{Lw}{T} = C \tag{24.11}$$

If the air is at first unsaturated, its partial potential temperature remains constant up to the condensation level. Above the condensation level, (24.11) holds. Lifting the air further the water vapor condenses and falls out until $w = 0$ when $p = 0$. The partial potential temperature, which is then attained, is the equivalent potential temperature

$$C = c_p \ln \Theta_E - AR \ln P \tag{24.2}$$

Upon combining (24.11) and (24.2), it is found that

$$\Theta_E = \Theta_d e^{\frac{Lw}{c_p T}} \tag{24.3}$$

or if the temperature is introduced again from (13.6),

$$\Theta_E = T \left(\frac{P}{p - e} \right)^{\frac{AR}{c_p}} e^{\frac{Lw}{c_p T}} \tag{24.31}$$

When the air whose equivalent potential temperature is to be computed is not saturated, the temperature at the condensation level has to be inserted in the exponents in (24.3) and (24.31) instead of the actual temperature.

It is sometimes convenient to use the "equivalent temperature" T_E, which is realized by bringing the air from the pressure zero dry adiabatically not to the standard pressure of 1000 mb but to its original partial pressure.

Consequently,

$$\frac{T_E}{\Theta_E} = \left(\frac{p - e}{P} \right)^{\frac{AR}{c_p}} = \frac{T}{\Theta_d} \tag{24.4}$$

or, with (24.3),

$$T_E = T e^{\frac{Lw}{c_p T}} \tag{24.5}$$

This definition of the equivalent potential and equivalent temperature has been given by Rossby.[1] It has the advantage that it shows clearly the conservative property of Θ_E with respect to saturated adiabatic changes.

There are other, slightly different definitions of T_E and Θ_E. According to (24.5), approximately

$$T_E - T = L \frac{w}{c_p} \tag{24.6}$$

[1] Rossby, C.-G., *Mass. Inst. Tech. Met. Papers*, **1, 3, 1932.**

This relation, which according to Rossby's definition is only an approximation, has been taken as the definition of the equivalent temperature by Robitzsch[1] who followed some earlier ideas of von Bezold.[2] Robitzsch's definition states that the equivalent temperature is the temperature which the air assumes when all its water vapor condenses at constant pressure and the latent heat of condensation is released. Rossby has pointed out that it is impossible to visualize this process physically.

Robitzsch defines further as the equivalent potential temperature the equivalent temperature that the air assumes when it is brought dry adiabatically from its original pressure to the standard pressure of 1000 mb. It follows as the definition equation that

$$\Theta_E = T_E\left(\frac{P}{p}\right)^{\kappa} \tag{24.7}$$

This equation may be regarded as a simplification of (24.31).

Because Rossby's and Robitzsch's definitions are different, the values for Θ_E and T_E are also different. For example, let $p = 1000$ mb, $T = 302°$, and $w = 14 \times 10^{-3}$, so that the air is just saturated. Then the difference between T_E and T would be 36.1°, according to Rossby's formula (24.5), and 34.2°, according to Robitzsch's formula (24.6).

Petterssen[3] retains the names "equivalent temperature" and "equivalent potential temperature" for Robitzsch's definition and refers to Rossby's as the "pseudo-equivalent" and "potential pseudo-equivalent temperature," to indicate that the changes of air are pseudo-adiabatic. Stüve[4] calls the quantities as defined by Rossby "pseudotemperature" and "pseudopotential" temperature. A very exhaustive discussion of the possible definitions and their conservative properties has been given by Bleeker.[5] In practice the differences are mostly quite negligible.

[1] Robitzsch, M., *Met. Z.*, **45**, 313, 1928.

[2] Von Bezold, W., "Gesammelte Abhandlungen," Vol. 10, Friedrich Vieweg & Sohn, Brunswick, 1906.

[3] Petterssen, S., "Weather Analysis and Forecasting," p. 24, McGraw-Hill Book Company, Inc., New York, 1940.

[4] Stüve, G., "Handbuch der Geophysik," Vol. 9 (2), p. 238, Gebrüder Borntrager, Berlin, 1937.

[5] Bleeker, W., *Quart. J. Roy. Met. Soc.*, **65**, 542, 1939.

25. Wet-bulb Temperature and Wet-bulb Potential Temperature. The atmospheric humidity is measured either by the hair hygrometer or by the psychrometer. The psychrometer consists of an ordinary thermometer showing the temperature of the air and another thermometer whose bulb is kept covered with a thin film of water. The temperature of the wet-bulb thermometer has some important thermodynamical implications which will be discussed now.

The following notations are used:

T = temperature
T_w = temperature at the wet-bulb
w = mixing ratio
w_w = mixing ratio at the wet bulb
e = water-vapor pressure
e_w = water-vapor pressure at the wet bulb
L = latent heat of condensation at T
L_w = latent heat of condensation at T_w
c_p = specific heat at constant pressure of dry air
c_{pw} = specific heat at constant pressure of water vapor

It will be assumed, following Normand,[1] that the air in contact with the wet bulb is cooled by evaporation from the wet bulb so that saturation occurs at the temperature T_w indicated by the wet thermometer. This assumption implies that $(1 + w)$ gm of moist air are cooled in contact with the wet bulb from T to T_w, so that an amount of heat is released sufficient to produce $(1 + w_w)$ gm of saturated air. It follows that

$$L_w(w_w - w) = (c_p + c_{pw}w)(T - T_w) \tag{25.1}$$

If the vapor pressure is introduced by (5.8), (25.1) becomes approximately, because $c_{pw}w \ll c_p$,

$$e_w - e = \frac{c_p}{0.621L_w} p(T - T_w) \tag{25.11}$$

This equation may be used to find the vapor pressure and the relative humidity as functions of the difference between the dry and the wet thermometer. It is referred to as the "psychrometer equation."

[1] Normand, C. W. B., *Mem. Ind. Met. Dept.*, **23**, 1, 1921.

Equation (25.1) may be written in the form

$$(c_p + c_{pw}w)T + L_w w = (c_p + w_w c_{pw})T_w + L_w w_w - c_{pw}(w_w - w)T_w \quad (25.2)$$

This relation states *Normand's first proposition* that, apart from the small difference between L_w and L, the heat content of the air is equal to the heat content of the same air at the wet-bulb temperature minus the heat content of the water vapor necessary to saturate it.

The air becomes saturated when it reaches the wet-bulb temperature. It can therefore not be cooled below the wet-bulb temperature by evaporation.

The process that is assumed to take place at the wet bulb is irreversible, for heat is taken away from the air at a temperature between T and T_w and added to the water at a temperature T_w where it is used to evaporate the water. Therefore, the entropy of the system increases. The system consisted at first of $(1 + w)$ gm of moist air at the temperature T and $(w_w - w)$ gm of water at the temperature T_w. Finally, there are $(1 + w_w)$ gm of saturated air at the temperature T_w. According to the definition of entropy (22.2), the gain of entropy by the water is

$$(c_p + wc_{pw})\,\frac{T - T_w}{T_w}$$

and the loss of the entropy by the air is

$$(c_p + wc_{pw}) \int_{T_w}^{T} \frac{dT}{T} = (c_p + wc_{pw}) \ln \frac{T}{T_w}$$

Because

$$\ln \frac{T}{T_w} = \frac{T - T_w}{T_w} - \frac{1}{2}\left(\frac{T - T_w}{T_w}\right)^2 + \cdots$$

the net gain of entropy is

$$(c_p + wc_{pw}) \left[\frac{1}{2}\left(\frac{T - T_w}{T_w}\right)^2 \pm \text{terms of higher order}\right]$$

This is approximately $\frac{1}{2}(T - T_w)/T_w$ times the net gain of the entropy by the water. Because $(T - T_w)/T_w$ is a small quantity, the gain of entropy of the whole system can be neglected in comparison with the loss of entropy by the air and the gain of

entropy by the water. Hence, *Normand's second proposition* follows, according to which the entropy of air is approximately equal to the entropy of the same air when saturated at the wet-bulb temperature minus the entropy of the water required to saturate it.

If saturated air of temperature T_w and mixing ratio w_w ascends adiabatically, it follows a pseudo-adiabat. In Fig. 13 the point B represents the original state of the air, and BD the pseudo-adiabat followed during the ascent. The amount of condensed water, say $w_w - w$, falls out and the mixing ratio decreases to w. If the air is brought down adiabatically to its original pressure level, it follows the dry adiabat DA. At A the temperature of the air may be T. Its mixing ratio w remains constant

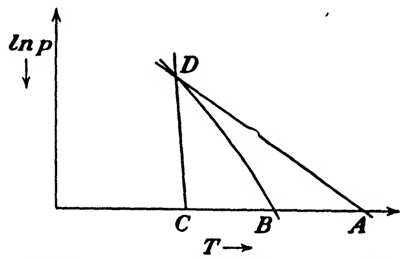

FIG. 13.—Relation between the temperature and the wet-bulb temperature.

during the descent. The process represented by ADB is not reversible for the condensed water has been expelled. Apart from the heat of condensation, no heat has been added or taken away from the air. Thus, the entropy of the moist air at A is equal to the entropy at B minus the entropy of the precipitated water. Therefore, according to Normand's second proposition the temperature T_w of the saturated air at B is the wet-bulb temperature of the air whose temperature and mixing ratio are T and w, respectively, at A. Because the air ascending from A becomes saturated at D, this point is the condensation point. The line DC may represent the saturation mixing-ratio line (dew-point line) through the point D. It has been shown in Sec. 13 that the dew point moves along the saturation mixing-ratio line during dry-adiabatic processes. Therefore, the point C where DC intersects the isobar through A is the dew point of the air whose temperature is T and whose mixing ratio is w. Consequently the dry adiabat through the temperature, the pseudo-

adiabat through the wet-bulb temperature T_w, and the saturation mixing-ratio line through the dew point meet in the condensation point. This is referred to as *Normand's third proposition.* Because Normand's second proposition is not strictly accurate and because the condensation of the water vapor on the saturated adiabat *BD* occurs at temperatures different from T_w, Normand's third proposition is not quite accurate,[1] or, to put it differently, the wet-bulb temperature determined with the aid of Normand's third proposition is not identical with the one defined by (25.1). Bleeker has therefore suggested the name "adiabatic wet-bulb temperature" when Normand's third proposition is accepted as the definition of the wet-bulb temperature. Petterssen[2] prefers the term "pseudo-wet-bulb temperature" in order to indicate that the changes of the air are pseudo-adiabatic. In practice the difference between the wet-bulb temperature and the pseudo-, or adiabatic, wet-bulb temperature is appreciable only if $T - T_w$ is large, *i.e.*, at high temperatures and low humidities. Actually, however, this difference is negligible.

The wet-bulb temperature remains constant when evaporation or condensation takes place at constant pressure. When the air undergoes adiabatic changes in the saturated or unsaturated stage, the wet-bulb temperature follows always the same pseudo-adiabatic curve. We may therefore define the wet-bulb potential temperature Θ_w as the wet-bulb temperature that the air assumes when it is brought adiabatically to the standard pressure. Θ_w is then invariant with respect to dry- or moist-adiabatic changes and to evaporation at constant pressure; it is another conservative property of the air. More accurately, one should distinguish between the potential wet-bulb temperature and the potential adiabatic, or potential pseudo-wet-bulb temperature. The former is strictly conservative with respect to evaporation, the latter with respect to dry- or moist-adiabatic changes, as follows from the foregoing discussion.

The (potential pseudo) wet-bulb potential temperature and the (potential pseudo) equivalent potential temperature stand in a direct functional relationship so that one can be computed when the other is given. To find this relation, consider Eq. (24.2). When the air is first lifted adiabatically to the pressure 0,

[1] Bleeker, W., *Quart. J. Roy. Met. Soc.*, **65**, 542, 1939.

[2] Petterssen, *op. cit.*, p. 25.

so that all the water vapor condenses and falls out, and then brought down to the standard pressure of 1000 mb, the temperature becomes Θ_E for which (24.2) holds. On the other hand, when the air is brought moist adiabatically to the standard pressure the wet-bulb potential temperature Θ_w is realized for which

$$c_p \ln \Theta_w + \frac{Lw(\Theta_w)}{\Theta_w} - AR \ln P = C \tag{25.21}$$

according to (24.1). For w the saturation value at the temperature Θ_w is to be taken. Because the constant is the same in (24.2) and (25.21), it follows that[1]

$$\Theta_E = \Theta_w e^{\frac{Lw(\Theta_w)}{c_p\Theta_w}} \tag{25.3}$$

A similar relation holds between the equivalent temperature and the wet-bulb temperature. If the exponential in (25.3) is developed in a series and terms of higher order are omitted, it is found that, approximately,

$$c_p\Theta_E = c_p\Theta_w + Lw(\Theta_w) \tag{25.31}$$

Thus, the equivalent potential temperature is approximately the temperature that dry air must have at the standard pressure for its heat content to equal the heat content of the air at the wet bulb at standard pressure.

The application of the wet-bulb potential temperature to air-mass analysis has been studied extensively by Hewson.[2] Because the wet-bulb potential temperature and the equivalent potential temperature are determined by each other according to (25.3), either quantity may be used in practical work. In fact, it may be said that the choice between the wet-bulb potential temperature and the equivalent potential temperature amounts only to a decision as to whether one prefers to mark the pseudo-adiabats by the temperature of the isotherms that they intersect at the pressure of 1000 mb or by the temperature of the dry adiabats that they reach at the pressure 0 mb. (This is strictly correct only for the potential pseudo wet-bulb temperature and the potential pseudo-equivalent temperature, in

[1] BINDON, H. H., *Monthly Weather Rev.*, **68**, 243, 1940.

[2] HEWSON, E. W., *Quart. J. Roy. Met. Soc.*, **62**, 387, 1936; **63**, 7, 323, 1937; **64**, 407, 1938.

Petterssen's terminology.) A practical advantage of the wet-bulb potential temperature over the equivalent potential temperature is that it can be determined more easily from the adiabatic or any other thermodynamic chart (see Sec. 28), for it does not require that the air be lifted to the pressure 0 mb.

26. Latent Instability. Frequently the lapse rate of temperature is intermediate between the dry and the moist adiabatic so that the atmosphere is conditionally unstable (see Sec. 20). Consider such a conditionally unstable temperature distribution, as shown by the full curve *ACED* in Fig. 14. If the air at *A* is unsaturated, it follows the dry adiabat *AB* when lifted adiabatically until it becomes saturated at *B*. From there on, it

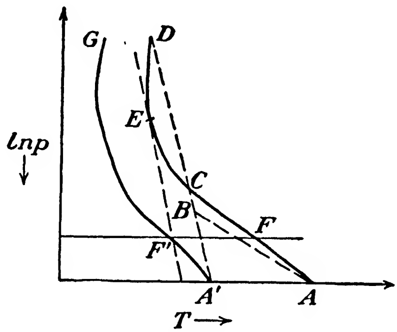

FIG. 14.—Latent instability.

cools at the saturated-adiabatic rate represented by the line *BCD*. At *C*, it is again in equilibrium with the air surrounding it. Up to this point, energy is required to lift the air, for it is cooler than the surrounding air. But from *C* upward the ascending parcel of air would be warmer than the surrounding air. It can therefore now rise spontaneously until it again attains equilibrium with the environment at *D*. This type of conditional instability which can be released by vertical displacement of a parcel of air has been called *latent instability* by Normand.[1]

Normand distinguishes between two cases of latent instability: (1) If the positive area *CDE* is larger than the negative area *ABC*, there will be a net gain of energy that will partly be used to overcome friction and that may partly be changed into kinetic energy. This case is referred to as *real latent instability*. (2) If the positive area *CDE* is smaller than the negative area *ABC*,

[1] NORMAND, C. W. B., *Quart. J. Roy. Met. Soc.*, **64**, 47, 1938.

there is a net loss of energy. This is the case of *pseudo-latent instability*. In the latter case, convection is much less likely to arise than in the former. In both cases the initial stability represented by the lower, negative area has, of course, to be overcome first.

Latent instability can exist only where the stratification is conditionally unstable, but it does not always prevail under conditionally unstable conditions. To find a criterion for latent instability, consider the wet-bulb temperature curve $A'F'G$ in Fig. 14 which belongs to the temperature distribution $AFCED$. The wet-bulb temperature corresponding to A is represented by A', which lies on the pseudo-adiabat through the condensation point B of A and on the isobar $A'A$ through A. It will be seen that latent instability exists for a given parcel of air if the pseudo-adiabat through its wet-bulb temperature intersects the curve representing the atmospheric stratification. Thus, the wet-bulb curve affords an easy method of finding the regions of latent instability. Above the pressure level indicated by the isobar FF', no latent instability exists because the pseudo-adiabat through the corresponding point F' on the wet-bulb curve just touches the curve showing the temperature distribution at the point E. Nevertheless, the atmosphere is conditionally unstable up to E.

It must be emphasized that these stability considerations do not take into account the fact that downward motions must take place in the surrounding air in order to satisfy the continuity conditions. The effects of these compensation motions on the stability have been studied by J. Bjerknes[1] and by Petterssen.[2]

27. Potential, or Convective, Instability. When vertical motion affects the whole layer of air, instead of a small parcel of air, the lapse rate of the air undergoing a vertical displacement is changed so that the stratification may become unstable when it was stable to begin with.

The analytical expression for the variation of the lapse rate of a layer subjected to a dry-adiabatic change is represented by (10.2). To derive a similar formula for air in the saturated

[1] Bjerknes, J., *Quart. J. Roy. Met. Soc.*, **64**, 325, 1938.

[2] Petterssen, S., *Geofys. Pub.*, **12**, No. 9, 1939; "Weather Analysis and Forecasting," p. 64.

state would lead to rather complicated formulas. The problem can be solved readily, however, with the aid of a pseudo-adiabatic chart. As an example, consider a layer whose pressure, temperature, and mixing ratio are, respectively, at the bottom (point A) 1000 mb, 15°C, and 9 gm/kg and at the top (point B) 900 mb, 12°C, and 4.0 gm/kg (Fig. 15). When the layer is lifted without any change of its cross section, the pressure difference between bottom and top must remain constant owing to the conservation of mass. When A ascends to the pressure level 800 mb, it follows first the dry adiabat to the condensation point C and then the

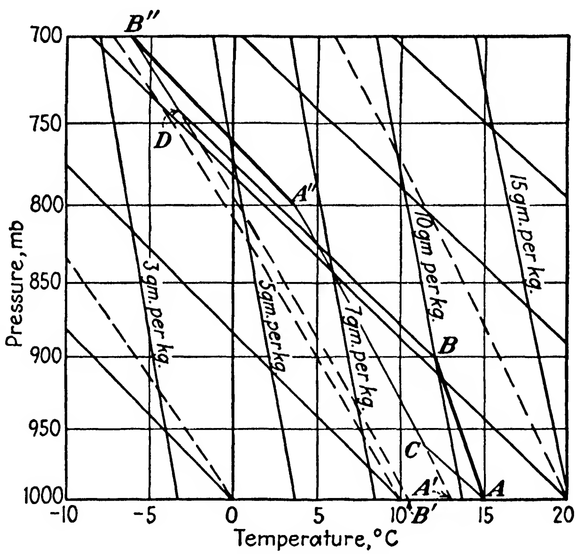

FIG. 15.—Convective or potential instability.

moist adiabat to A''. Similarly, B, when lifted 200 mb, follows the dry adiabat to the condensation point D and then the moist adiabat to B'' where the pressure is 700 mb. Originally, the lapse rate of the layer AB was less than moist adiabatic; in fact, the layer was not even conditionally unstable. After it has been lifted 200 mb, its lapse rate is greater than the saturated adiabatic. Because the air column has become saturated, it is now unstable. The reader may verify on a thermodynamic chart the fact that the lapse rate of the layer becomes greater than dry adiabatic when the layer is lifted another 150 mb. This type of instability, where the layer was

stable originally but becomes unstable owing to vertical lifting, is called *potential*[1] or *convective instability.*[2]

The development of instability during continued lifting is due to the unequal heating of the air originally at A and at B by the latent heat of condensation. The amount of heat supplied to the air originally at A is larger than the amount of heat supplied to the air originally at B, for the air at A reaches the condensation level earlier than the air at B. When the moist adiabats through A'' and B'', which of course also pass through the respective condensation points C and D, are extended to the 1000-mb isobar, the wet-bulb potential temperatures A' and B' are obtained. In the present example the wet-bulb potential temperature decreases with the altitude. It is easily seen that a layer is potentially unstable when its wet-bulb potential temperature Θ_w decreases with height. When the layer is lifted, the variations of the wet-bulb temperature T_w of any particle in the layer follow a saturated adiabat. As soon as the air becomes saturated, its temperature and wet-bulb temperature coincide. When the lapse rate of T_w was originally larger than the saturated adiabatic, it remains always larger during the lifting, and the layer becomes therefore unstable after saturation. A lapse rate of T_w greater than the saturated adiabatic implies a decrease of Θ_w with the elevation. It follows that $d\Theta_w/dz < 0$ is the condition for potential instability.

Since the wet-bulb potential temperature Θ_w and the equivalent potential temperature Θ_E are related by (25.3), the condition for potential instability may also be written $d\Theta_E/dz < 0$.

28. Thermodynamic Charts and Air-mass Charts. According to Sec. 21 the area enclosed by the curve that represents a cyclic process on an indicator diagram measures the amount of energy transformed during the process. In meteorological practice the use of the specific volume v as the one coordinate is very unsatisfactory, for v cannot be observed directly. But on the adiabatic chart that has T and $\ln p$ as coordinates the energy involved in cyclic processes is also represented by an area according to Sec. 23. The work done or gained during the ascent of a parcel of air is given by the area between the curves representing the changes of state of the parcel and the actual

[1] HEWSON, E. W., *Quart. J. Roy. Met. Soc.*, **63**, 323, 1937.

[2] ROSSBY, C.-G., *Mass. Inst. Tech. Met. Papers*, **1**, 3, 20, 1932.

atmospheric stratification. It will be shown below that the adiabatic chart is an equal-area transformation of the pv diagram. Obviously, any diagram that can be obtained from the pv diagram by an equal-area transformation will serve as well. Such diagrams may be called "thermodynamic charts."

Hitherto, we have used only the adiabatic or pseudo-adiabatic chart because its coordinates T and p can be observed directly so that it may be regarded as the simplest thermodynamic chart in meteorology. Because a great variety of thermodynamic charts[1] are now employed in meteorology, it is useful to investigate the condition for equal area transformations of the pv diagram.[2] The coordinates of the new chart may be denoted by x and y. They may be regarded as functions of p and v,

$$x = x(p, v) \qquad y = y(p, v) \tag{28.1}$$

The condition for an equal-area transformation requires that

$$\frac{\partial x}{\partial p}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial p} = 1 \tag{28.2}$$

For a derivation of this condition, reference may be made to the textbooks on differential geometry.

Any pair of variables x and y that satisfies (28.2) can serve as new coordinates. In practice, when a thermodynamic chart is to be constructed, one of the coordinates will in general be known, whereas the other is to be found. If $x = x(p, v)$ is known, then $M = \partial x/\partial v$ and $N = \partial x/\partial p$ are also given. From (28.2), it follows that y is the solution of the partial differential equation

$$N\frac{\partial y}{\partial v} - M\frac{\partial y}{\partial p} = 1 \tag{28.21}$$

To solve (28.21), consider the system of simultaneous ordinary differential equations

$$\frac{dv}{N} = -\frac{dp}{M} = \frac{dy}{1} \tag{28.22}$$

[1] For a complete collection and description of the charts see L. Weickmann, "Ueber aerologische Diagrammpapiere," Internationale Organisation, Internationale Aerologische Kommission, Berlin, 1938.

[2] Refsdal, A., *Geofys. Pub.*, **11**, No. 13, 1937. Werenskiold, W., *Geofys. Pub.*, **12**, No. 6, 1938.

The equation $\frac{dv}{N} = -\frac{dp}{M}$ is satisfied by the relation

$$x(p, v) = x_0 = \text{const}$$

The variable v may now be expressed as a function of p and x_0 so that M appears as a function of the variable p only. Then y may be found by integration of the equation

$$-\frac{dp}{M} = \frac{dy}{1}$$

according to which

$$y = -\int \frac{dp}{M} + F(x) \tag{28.3}$$

$F(x)$ is an arbitrary function of x. After the integration is performed, x_0 has to be replaced again by x in (28.3).

For instance, if

$$x = v^\alpha p^\beta \tag{28.4}$$

it follows that

$$M = \frac{\partial x}{\partial v} = \alpha v^{\alpha-1} p^\beta$$

Substituting here from the equation

$$x_0 = v^\alpha p^\beta$$

$$M = \alpha x_0^{\frac{\alpha-1}{\alpha}} p^{\frac{\beta}{\alpha}}$$

Upon introducing this expression for M in (28.3), integrating, and writing again $v^\alpha p^\beta$ for x_0, it is found that

$$y = -\frac{1}{\alpha - \beta} v^{1-\alpha} p^{1-\beta} \tag{28.41}$$

where it has been assumed that $F(x) = 0$. When $\alpha = 0$ and $\beta = \kappa = AR/c_p$, one of the coordinates becomes p^κ, which is often used as one of the coordinates, and the temperature T as the other coordinate (see Sec. 9). From (28.41), it follows that, in the case of an equal-area transformation,

$$y = \frac{1}{\kappa} v p^{1-\kappa}$$

or, with Eq. (9.1),

$$y = \frac{R}{\kappa} P^{-\kappa} \Theta$$

where P is the standard pressure of 1000 mb and Θ the potential temperature. Thus the chart with the coordinates p^κ and T is not an equal-area transformation of the pv diagram and cannot be used for energy computations, although the errors would in general only be slight.

The case $\beta = \alpha$ requires a separate treatment. It is found that

$$y = -\frac{1}{\alpha} (pv)^{1-\alpha} \ln p \qquad (28.42)$$

When $\alpha = \beta = 1$,

$$x = pv = RT \qquad \text{and} \qquad y = -\ln p$$

This is the *adiabatic* or *pseudo-adiabatic chart* which has been used hitherto.

As another example, let

$$x = \alpha \ln v + \beta \ln p \qquad (28.5)$$

Then

$$y = -\frac{pv}{\alpha - \beta} = -\frac{RT}{\alpha - \beta} \qquad (28.51)$$

When $\alpha = c_p$, and $\beta = c_v$ according to (28.51) and (22.44)

$$\begin{aligned} x &= c_p \ln v + c_v \ln p \\ &= c_p \ln T - (c_p - c_v) \ln p + c_p \ln R \\ &= \phi + \text{const} \end{aligned}$$

and

$$y = -\frac{T}{A}$$

Shaw[1] has constructed such a chart with the absolute temperature T as abscissa, and the entropy of dry air ϕ as ordinate. This chart is called the *tephigram*. Because the entropy is proportional to the logarithm of the potential temperature according

[1] SHAW, N., "Manual of Meteorology," Vol. 3, University Press, Cambridge, p. 269.

to (22.5), the ordinate gives also the potential temperature on a logarithmic scale. The dry adiabats are thus horizontal, a circumstance useful for practical work.

When $\alpha = \beta$ in (28.5), it is found that

$$y = -\frac{RT}{\alpha}\ln p \tag{28.52}$$

In particular, if $\alpha = \beta = 1$,

$$x = \ln RT \qquad \text{and} \qquad y = -RT\ln p$$

A thermodynamic chart with these coordinates, the *aerogram*, has been constructed by Refsdal.[1] It has a great number of advantages although the many lines may be somewhat disturbing when it is used in practical work. The height difference between two pressure levels is represented by the length of the isotherm representing the mean temperature between the two isobars on the aerogram. This permits a very quick computation of the height. For a proof of this theorem and for a detailed description the reader is referred to Refsdal's original paper. Similar graphical methods of height computation can be adopted on other thermodynamic charts as pointed out by Spilhaus.[2] In an atmosphere with adiabatic lapse rate, by logarithmic differentiation of (9.1) and from (6.11),

$$\frac{dT}{T} = \kappa\frac{dp}{p} = -\kappa\frac{d\psi}{RT}$$

By integration between two levels 1 and 2, it follows that

$$\psi_1 - \psi_2 = \frac{R}{\kappa}(T_2 - T_1) \tag{28.6}$$

Thus, to determine the height difference between p_1 and p_2, the adiabat has to be found that encloses the same area between the isobars p_1 and p_2 as the actual temperature distribution. On thermodynamic charts with the temperature as one coordinate the projection of the adiabat between p_1 and p_2 on the isotherm is proportional to the height difference, and a scale can be constructed from which the height difference can be read off directly.

[1] Refsdal, A., *Geofys. Pub.*, **11**, No. 13, 1937.

[2] Spilhaus, A. F., *Bull. Am. Met. Soc.*, **21**, 1, 1940.

In addition to the thermodynamic charts that are equal-area transformations of the pv diagram, other charts have been constructed, not intended by their authors to be used for height and energy computations. When aerological ascents are plotted on these charts, certain features of the vertical stratification are stressed so that these charts are particularly useful for certain purposes of air-mass analysis. To indicate that these charts are of a different type insofar as they are not equal-area transformations of the pv diagram, they may be called "air-mass charts"[1] even though the charts described earlier also permit, of course, an analysis of the structure of the atmosphere.

Rossby has constructed a chart[2] with the mixing ratio w as abscissa and the partial potential temperature on a logarithmic scale as ordinate. Because

$$w = 0.621 \frac{e}{p - e} \tag{5.8}$$

and

$$\Theta_d = T\left(\frac{P}{p - e}\right)^{\frac{AR}{c_p^*}} \tag{13.6}$$

and because e, the saturated vapor pressure, is only a function of the temperature, the pressure p and the temperature T of the air are determined when w and Θ_d are given. If the air is not saturated, it may be lifted adiabatically to the condensation level without changing w and Θ_d. The values p and T obtained in the unsaturated case are therefore the pressure p_C and T_C at the condensation level. Because w, Θ_d, and T_C are thus determined by the position of a point on the chart, the equivalent potential temperature Θ_E can be computed from (24.3) for each point, and lines of equal equivalent potential temperature may be drawn. The diagram is therefore called the *equivalent-potential-temperature diagram* (or, more briefly, the *Rossby diagram*). The mixing ratio and the partial potential temperature of unsaturated air do not change during adiabatic variations. The air is therefore represented by one and the same characteristic point on the diagram until condensation occurs. When a layer of dry unsaturated air is thoroughly stirred so that it becomes homo-

[1] Weickmann, *loc. cit.*

[2] Rossby, C.-G., *Mass. Inst. Tech. Met. Papers* **1, 3**, 1932.

geneous, its potential temperature and mixing ratio will approach constant values, and the layer is shown as a very short line or, in the limiting case, as a point. On the other hand, where two air masses of very different properties meet, the transitional zone between these air masses is stretched over a long distance on the diagram. These properties and the possibility of finding the conservative equivalent potential temperature directly from the diagram make it a very useful tool for air-mass analysis.

Another air-mass chart that may be mentioned here is the *thetagram* developed by Schinze.[1] The abscissa of the diagram is the temperature and the ordinate the pressure, both on a linear scale. To facilitate the identification of air masses the thetagram shows furthermore the typical (mean) vertical distribution of the equivalent potential temperature as a function of the pressure in the various unmodified air masses. Because this element varies from month to month, different thetagrams have to be used for different months and the chart for each month shows the typical distribution of the equivalent potential temperature at the beginning and at the end of each month. Obviously, different thetagrams should be used also for regions of very different geographical location, *e.g.*, for Europe and North America.

Problem

7. A parcel of air has a temperature T' different from that of the air surrounding it, T. The temperature of the surrounding air decreases with the altitude at the constant lapse rate α. Find the motion of the parcel of air, neglecting the effects of a compensating motion of the surrounding air and assuming dry-adiabatic changes of state. Under which conditions is the motion stable and unstable? What are the period and amplitude of the oscillation in the stable case?

[1] SCHINZE, G., *Beitr. Phys. Atm.*, **19**, 79, 1932.

CHAPTER V

RADIATION

29. The Laws of Radiation. For an understanding of the effects of solar, terrestrial, and atmospheric radiation on the atmospheric processes a knowledge of the laws of radiation is necessary. These laws will be briefly recapitulated here, but for details the reader is referred to textbooks on the subject.

If the emissive power of a radiating body for a given wave length λ is e_λ and its absorptive power for the same wave length is a_λ, *Kirchhoff's law* states that the ratio of the two depends not on the nature of the body but only on the wave length of the radiation and the temperature of the body,

$$\frac{e_\lambda}{a_\lambda} = E_s(\lambda, T) \tag{29.1}$$

If the absorptive power $a_\lambda = 1$ for all wave lengths, *i.e.*, if a body absorbs all incident radiative energy, it is called a "black body." Thus, the function $E_s(\lambda, T)$ in (29.1) is the radiation of the black body. When the absorptive power a_λ of a radiating body is known, its emission is the fraction a_λ of the black-body radiation. Strictly "black" bodies do not exist in nature, but the radiation from the earth's surface or from a cloud surface, for instance, may be treated as black-body radiation.

If $a_\lambda = \text{const} < 1$ for all wave lengths, the radiation emitted is that of a "gray" body. Such gray bodies do not exist in nature, either; but they are a helpful device in the discussion of some radiation problems.

The energy distribution in the spectrum of a black body of the temperature T is given by *Planck's law*,

$$E_\lambda \, d\lambda = \frac{c_1 \lambda^{-5}}{e^{\frac{c_2}{\lambda T}} - 1} \, d\lambda \tag{29.2}$$

The constants c_1 and c_2 follow from the quantum theory.

At a fixed temperature T the radiative energy of a black body E_λ has a maximum for a certain wave length λ_{max} that can be found by differentiation of (29.2) with respect to λ and equating $dE/d\lambda$ to zero. The result is *Wien's law*,

$$\lambda_{max} = \frac{a}{T} \tag{29.3}$$

where a is a numerical constant whose value is 0.2892 cm deg.[1] Thus, the greater the temperature of a black body, the smaller the wave length of maximum intensity. In the visible part of the spectrum, this law leads to the well-known phenomenon that the light of a radiating body appears whiter, the hotter the body.

To obtain the total radiation intensity I of the black body at a given temperature T, Eq. (29.2) has to be integrated over the whole spectrum. It follows that

$$I = \sigma T^4 \tag{29.4}$$

This relation is known as *Stefan-Boltzmann's law.* The constant σ is called "Stefan's constant,"

$$\sigma = 5.70 \times 10^{-5} \text{ erg cm}^{-2} \text{ sec}^{-1} \text{ deg}^{-4}$$
$$= 0.817 \times 10^{-10} \text{ cal cm}^{-2} \text{ min}^{-1} \text{ deg}^{-4}.$$ [1]

Historically, the formulation of Stefan-Boltzmann's law and Wien's law precedes the formulation of the more general law of Planck.

Figure 16 shows the energy distribution in the spectrum of black bodies at some temperatures that may occur on the surface of the earth or in the atmosphere. The width of each band of the spectrum for which the radiation is given has been chosen equal to 1μ ($= 10^{-4}$ cm). It can be seen that the wave length of maximum energy emission is displaced toward longer wave lengths with decreasing temperature and that the total radiation energy decreases with decreasing temperature.

30. The Solar Radiation. The intensity of the solar radiation at the outer limit of the atmosphere on a surface perpendicular to the solar beam is 1.94 cal/cm^2 min^{-1} when the earth is at its mean distance from the sun. This value of the solar radiation

[1] WENSEL, H. T., *Bur. Standards, J. Research,* **22,** 375, 1939.

is called the "solar constant." The expression "constant" means not that the solar output of energy is invariable but only that the large apparent variations of the solar radiation have been eliminated that are due to variations of the transparency of the earth's atmosphere and to the changing distance between sun and earth. It is still an open question whether the remaining slight changes of the solar constant are caused by real variations of the sun's radiation intensity or by an incomplete elimination of the effects of the earth's atmosphere.[1]

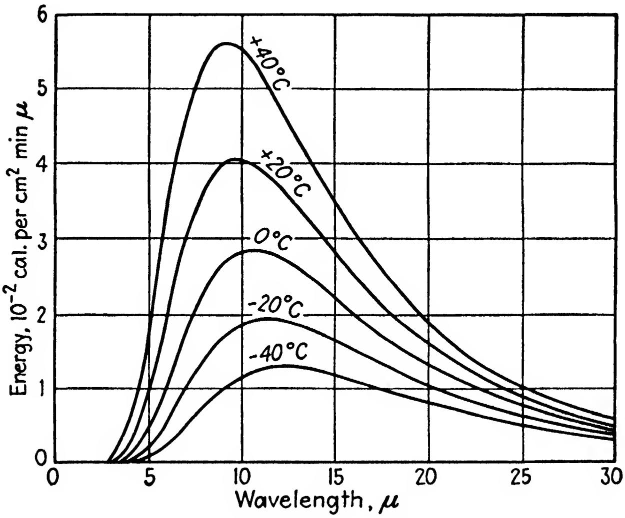

Fig. 16.—Black-body radiation at different temperatures.

When the solar constant is known, it can be computed from (29.4) what temperature a black body of the same size and at the same distance as the sun should have in order to emit the same amount of radiation. This quantity is called the "effective" temperature. The effective temperature of the sun is found to be 5760° abs.

Another method of arriving at a figure for the sun's temperature is given by Wien's law (29.3). Outside the earth's atmosphere the energy maximum of the solar spectrum is at the wave

[1] See numerous papers by C. G. Abbott and collaborators mainly in *Smithsonian Misc. Coll.* Also, *Ann. Astrophys. Obs. Smithsonian Inst.*, **2,** 1908; **3,** 1913; **4,** 1922; **5,** 1932. Paranpje, M. M., *Quart. J. Met. Soc.*, **64,** 459, 1938.

length 0.475μ. It follows that the temperature of the sun according to Wien's law, the so-called "color" temperature, is 6090° abs. The difference between the effective temperature and the color temperature of the sun is not surprising. Even though the photosphere of the sun may radiate as a black body, the absorption and radiation in the sun's outer atmosphere modify the character of the radiation finally leaving the sun.[1] The temperature of about 6000° abs is of course far lower than the temperatures that must exist in the interior of the sun.

The solar spectrum is not continuous but interrupted by many dark lines and bands in which very little or no energy emission takes place. A great number of these lines are due to absorption of the radiation in the outer atmosphere of the sun. Other lines and bands, however, are caused by the absorption of radiation in the earth's atmosphere.

31. The Geographical and Seasonal Distribution of the Solar Radiation in the Absence of the Atmosphere. The total amount of solar radiation received by the earth is obtained by multiplying the solar constant by the cross section of the earth πE^2. To find the mean value of the solar radiation over the whole earth, this last figure has to be divided by the area of the earth

$$J_m = \frac{\pi E^2}{4\pi E^2} J_0 \tag{31.1}$$

where J_0 is the solar constant. Of course, the actual values of the solar radiation received deviate considerably from J_m with the season and the geographical latitude and the varying distance between earth and sun. The computation of these figures is an astronomical rather than a meteorological problem.[2] Figure 17 shows the daily amount of solar radiation that would be received per square centimeter of the horizontal surface of the earth, in the absence of the atmosphere, at different latitudes and at different times of the year. The ordinate of this figure is the geographic latitude, and the abscissa the time of the year expressed by the longitude of the sun on the upper margin and by the corresponding date on the lower margin. At the summer solstice the North Pole receives the maximum amount of radiation; for the sun is

[1] Milne, E. A., *Monthly Not. Roy. Astr. Soc.*, **81**, 375, 1912.

[2] Milankovitch, M., in Koeppen-Geiger, "Handbuch der Klimatologie," Vol. 1, A, Gebrüder Bornträger, Berlin, 1930.

here shining during 24 hours of the day, the obliqueness of the sun's radiation being thus more than compensated for. A secondary maximum, indicated only by a bulging of the line 1000 cal/cm^2 day, exists at 40° north latitude. In the south-polar regions, no solar radiation is received, for the sun is always below the horizon during the northern summer. During the winter season the distribution of the solar radiation is reversed. The radiation received by the Southern Hemisphere during the north-

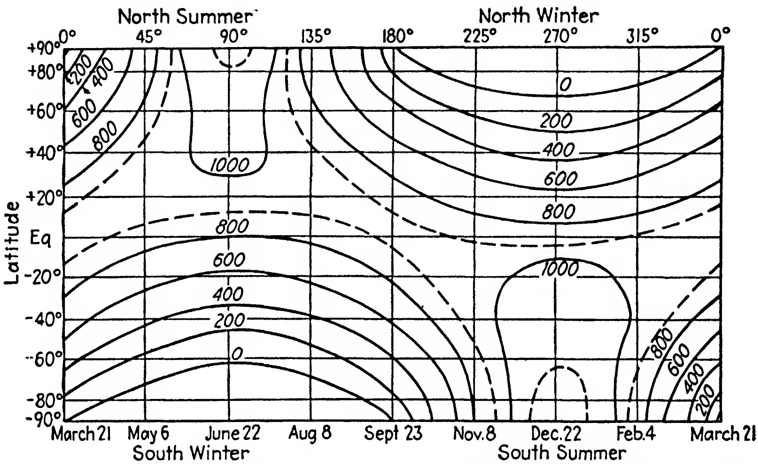

FIG. 17.—Daily insolation in calories per square centimeter received at the earth's surface in the absence of the atmosphere. (*After Milankovitch's computations.*)

ern winter is larger than the radiation received by the Northern Hemisphere during its summer, for the earth is closer to the sun during the northern winter than during the northern summer.

32. The Depletion of the Solar Radiation in the Earth's Atmosphere. Owing to the presence of the atmosphere the intensity of the solar radiation reaching the earth's surface is smaller than as it is shown in Fig. 17. During the passage through the atmosphere a fraction of the solar radiation is absorbed by some of the atmospheric gases, water vapor, carbon dioxide, oxygen, and, in higher layers, ozone. The absorption lines and bands of terrestrial origin in the solar spectrum can be distinguished from those due to the absorption in the sun's atmosphere, for their intensity varies with the length of the path that the solar radiation has to travel through the earth's atmos-

phere. The terrestrial lines become stronger the larger the zenith distance of the sun. Furthermore, when the spectroscope is directed at the solar limb, the lines of solar origin show a Doppler effect due to the rotation of the sun, whereas the terrestrial lines show, of course, no such displacement.

In discussing the effect of atmospheric absorption on solar radiation on the one hand and on radiation from the ground and from the atmosphere on the other hand, it is important to know that the solar radiation resembles the black-body radiation at a temperature of about 6000° abs, whereas the terrestrial radiation is emitted at temperatures of 200 to 300° abs. Consequently, the radiative energy of the sun up to about 99 per cent is contained between the wave lengths 0.17μ and 4μ, with a maximum in the visible spectrum at 0.475μ. The radiative energy of a black body at a temperature of 300° abs is contained, up to 99 per cent, between about 3μ and 80μ, with a maximum at 10μ; that of a black body at 200° abs., between 4μ and 120μ, with a maximum at 15μ. Thus, the solar radiation may be said to be of much shorter wave length than the radiation from the surface of the earth or from the atmosphere.

The short wave end of the solar spectrum is terminated suddenly at about 0.3μ owing to the absorption of the atmospheric ozone[1] in this spectral region. The atmospheric ozone is mainly concentrated in regions between 20 and 30 km altitude. The ratio of the density of ozone to the density of air has a maximum at a higher level, at about 40 km, owing to the decrease of the air density with elevation. A discussion of the statistical relationship between ozone and meteorological variables, especially of the high correlation between the total amount of ozone and the potential temperature in the stratosphere, has been given by Meetham.[2]

The loss of incident solar radiation by the absorption in the atmosphere is small. It is mainly due to water vapor,[3] for the oxygen lines are narrow, and carbon dioxide does not absorb in the spectral region in which the main part of the sun's radiative

[1] Goetz, F. W. P., *Gerl. Beitr. Geophys.*, 3d suppl. vol. "Ergebnisse der kosmischen Physik," Vol. 3, p. 251, Akademische Verlagsgesellschaft, Leipzig, 1938.

[2] Meetham, R., *Quart. J. Roy. Met. Soc.*, **63**, 289, 1937.

[3] Fowle, F. E., *Astrophys. J.*, **42**, 409, 1915.

energy is concentrated. The heating of the atmosphere by direct solar radiation is therefore not important.

Apart from absorption the energy of the solar beam is depleted by scattering and diffuse reflection in the atmosphere. It was shown by Lord Rayleigh[1] that the diminution of the radiation intensity due to the scattering on the air molecules is inversely proportional to the fourth power of the wave length. Thus, light of short wave length is scattered more than light of long wave length. Consequently, in the scattered light from the sky the shorter bluish waves predominate, as indicated by the blue color of the sky. On the other hand, in the direct light coming from the sun, the components of shorter wave lengths become less intense the closer the sun is to the horizon so that the wave length of maximum intensity is displaced toward the red side of the solar spectrum.

For particles larger than air molecules the coefficient of scattering is inversely proportional not to the fourth but to a smaller power of the wave length.[2] When the particles are sufficiently large, the depletion becomes independent of the wave length, and the process of scattering is superseded by diffuse reflection. Because the diffuse reflection is the same for all wave lengths, the blueness of the sky is less pure the greater the number of large particles. These large particles are mainly dust, water droplets, and ice crystals, so that the blueness of the sky gives at least a qualitative indication of the amount of these atmospheric impurities.[3]

To obtain a rough measure for the depletion of the solar radiation during its passage through the earth's atmosphere, it may be assumed that the depletion is independent of the wave length. The following relations are therefore correct only for the radiation at a given wave length.

From the law of Bouguer and Lambert, it follows that the loss of radiation passing through a layer of the thickness dz

$$dJ = -k(z)J \sec \zeta \, dz \tag{32.1}$$

where J is the intensity of the radiation, k a proportionality

[1] See, for instance, W. Humphreys, "Physics of the Air," 2d ed., p. 538, McGraw-Hill Book Company, Inc., New York, 1929.

[2] Linke, F., and von dem Borne, H., *Gerl. Beitr. Geophys.*, **37**, 49, 1932.

[3] Ostwald, W., and Linke, F., *Met. Z.*, **45**, 367, 1928.

factor, and ζ the zenith distance (Fig. 18). The level above which the effect of the depletion vanishes may be at the altitude h. Here the intensity of the solar radiation is equal to the solar constant J_0 except for the varying distance of the sun. It follows by integration of (32.1) that the solar radiation at sea level, $z = 0$,

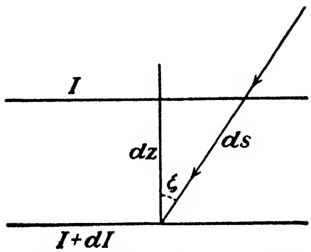

FIG. 18.—The law of Bouguer and Lambert.

$$J = J_0 e^{-\sec \zeta \int_0^h k(z)\,dz} \tag{32.2}$$

The quantity

$$q = e^{-\int_0^h k(z)\,dz} \tag{32.3}$$

is the "transmission coefficient." At a zenith distance ζ of the sun the solar radiation arriving at the surface of the earth is therefore

$$J = J_0 q^{\sec \zeta} \tag{32.4}$$

To obtain the amount of radiation received per unit area of the earth's surface, this value of J has to be multiplied by $\cos \zeta$.

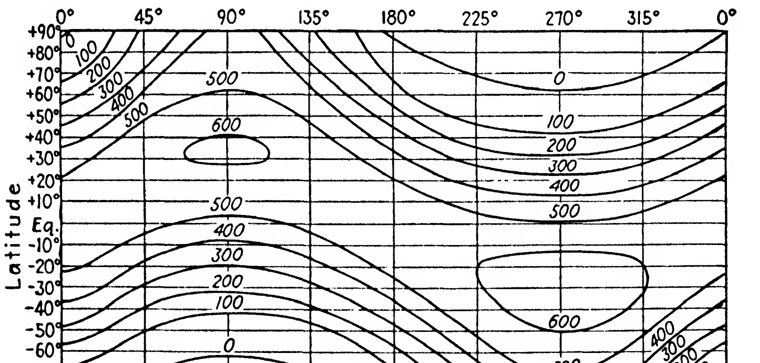

FIG. 19.—Daily insolation in calories per square centimeter received at the earth's surface when the atmospheric transmission coefficient is 0.7. (*After Milankovitch's computations.*)

Figure 19 shows the daily amount of solar radiation in calories per square centimeter at the earth's surface during the year

when the transmission coefficient is 0.7, according to computations by Milankovitch.[1] The daily totals are now considerably less than those shown in Fig. 17. The loss is particularly strong in the polar regions on account of the great zenith distance of the sun. In lower latitudes the loss by extinction in the atmosphere is less so that the summer maxima are now found approximately in the same position as the secondary maxima in the case of an earth without atmosphere (Fig. 17).

When the sun is near the horizon, the simple secant law has to be replaced by a more complicated one[2] on account of the curvature of the earth and the atmospheric refraction.

The transmission coefficient may be used as a measure of the effect of impurities suspended in the atmosphere, the atmospheric turbidity. To obtain better expressions for the atmospheric turbidity, Linke has defined a turbidity factor; Ångström, a turbidity coefficient. For a discussion of these quantities which is beyond the scope of this book the reader is referred to the literature.[3]

33. The Albedo of the Earth. The radiation that is scattered and reflected diffusely is partly directed back into space, perhaps after secondary scattering or reflection has taken place, so that it is lost to the terrestrial heat balance. In addition a certain amount of radiation is reflected at the ground and at cloud surfaces. The ratio of the radiative energy reflected and scattered back into space to the radiative energy received is called the "albedo" of the earth. It varies considerably for surfaces of different nature.[4] For clouds, it is 0.78, for freshly fallen snow 0.81 to 0.85, for fields only 0.14. For water the albedo increases with increasing zenith distance ζ of the sun, from 0.29 when $\zeta = 78°$ to 0.62 when $\zeta = 86°$.

[1] Milankovitch, *loc. cit.*

[2] Tables of this function: Linke, F., Meteorologisches Taschenbuch," II, Table 68, Akademische Verlagsgesellschaft, Leipzig, 1933; "Smithsonian Meteorological Tables," Table 100, 5th ed., Smithsonian Institution, Washington, D. C., 1931.

[3] Linke, F., *Beitr. Phys. Atm.*, **10**, 91, 1928. Ångström, A., *Geografiska Annaler*, **11**, 156, 1929; **12**, 375, 1930. Feussner, K., and Dubois, P., *Gerl. Beitr. Geophys.*, **27**, 132, 1930. Wexler, H., *Trans. Am. Geophys. Un.*, *14th Ann. Meeting*, p. 91, 1933.

[4] Conrad, V., in Koeppen-Geiger, "Handbuch der Klimatologie," Vol. 1, B, p. 23. Gebrüder Bornträger, Berlin, 1930.

The mean albedo for the earth depends among other factors on the amount of cloudiness. Assuming a mean cloudiness of 52 per cent, Aldrich[1] determined the albedo to 0.43. According to Ångström[2] the albedo A is related to the cloudiness C by the formula

$$A = 0.17 + 0.53C \tag{33.1}$$

Danjon[3] did not find a very clear relation of the albedo to the cloudiness or to other factors such as snow cover and vegetation, from the observation material available at present. The mean visual albedo is 0.39 according to Danjon. The albedo of the earth for short wave lengths is larger than for long wave lengths.

Because the energy lost owing to reflection and scattering does not enter into the terrestrial heat balance, the radiation incident at the outer limit of the atmosphere must be diminished by the amount reflected back to space, in order to obtain the actual amount of energy available for atmospheric processes.

Thus, the mean value J_m of solar radiation received on the average over the whole earth [see (31.1)] would become 0.276 cal cm^{-2} min^{-1} when Aldrich's value for the albedo is used.

In drawing Fig. 19 the loss of radiation energy due to the albedo has not been taken into account. Because this loss is largely due to reflection from clouds, Fig. 19 represents the amount of solar radiation when the sky is clear.

34. Absorption of Terrestrial Radiation. Although the solar radiation is only slightly absorbed by the atmosphere, a considerable percentage of the long-wave radiation emitted by the ground and by the atmosphere is reabsorbed in the atmosphere.

To obtain an expression for the absorption, consider a monochromatic beam of the radiation intensity I_λ. This beam may pass perpendicularly through a thin layer of thickness dz. When the density of the absorbing substance in this layer is σ, the absorption is proportional to $\sigma\,dz$. The product $\sigma\,dz$ is called the "optical thickness" or "optical mass" of the layer and will be denoted by du here. It follows that

$$dI_\lambda = -k_\lambda I_\lambda\,du \tag{34.1}$$

[1] ALDRICH, L. B., *Smithsonian Misc. Coll.*, **69**, No. 10, 1919.
[2] ÅNGSTRÖM, A., *Gerl. Beitr. Geophys.*, **15**, 1, 1926.
[3] DANJON, A., *Ann. Obs. Strassbourg*, **3**, No. 3, 1936.

k_λ is the absorption coefficient of the substance for radiation of the wave length λ. By integration,

$$I_\lambda = I_{\lambda_0} e^{-k_\lambda u} \tag{34.11}$$

This relation is known as *Beer's law.*

To determine the radiation of the atmosphere, one has to know the absorption coefficients of the atmospheric gases in the wave-length region from about 3μ to about 100μ where black-body radiation at terrestrial temperature takes place. Then the emission is also known according to Kirchhoff's law (29.1). The most important absorbing gas in the regions of the atmosphere in which we are interested is water vapor. Furthermore, absorption by carbon dioxide has to be taken into account.

In earlier investigations, it was assumed that water vapor may be treated as a gray absorber, *i.e.*, that it has a uniform absorption coefficient throughout the whole long-wave region. Investigations by Simpson[1] have shown that this method of attack is quite inadequate. The assumption of gray absorption leads to the conclusion that the radiation which is emitted by the earth and its atmosphere into space is practically independent of the surface temperature. Thus it would be impossible to see how the temperature of the air adjusts itself to changes in solar radiation.

It is therefore necessary to consider the variation of the absorption coefficient with the wave length. Until recently the absorption coefficients of water vapor mostly used in meteorology were those determined by Hettner[2] in steam. They show an intense band centered around 6.26μ and a wide band beginning at about 10μ and increasing with oscillations toward longer wave lengths. New measurements by Weber and Randall[3] at wave lengths larger than 10μ through moist air at room temperature have given lower values of the absorption coefficients, although they confirmed Hettner's results qualitatively.

In order to simplify the investigation of atmospheric radiation, Simpson[4] divided the atmosphere into layers each containing 0.3 mm (= 0.03 gm) of precipitable water per square centimeter cross section. If h is the height of a column of 1 cm^2 cross section

[1] Simpson, G. C., *Mem. Roy. Met. Soc.*, **2**, No. 6, 1928.
[2] Hettner, G., *Ann. Physik*, **55**, 476, 1918.
[3] Weber, L. R., and Randall, H. M., *Phys. Rev.*, **40**, 835, 1932.
[4] Simpson, C. G., *Mem. Roy. Met. Soc.*, **3**, No. 21, 1928.

containing a mm of precipitable water ($= a \times 10^{-1}$ gm), it follows that $10^{-1} a = h\rho_w$ where ρ_w is the density of water vapor. When the water-vapor pressure e is expressed in millibars and h in meters according to (5.1),

$$h = 4.62a \frac{T}{e} \tag{34.2}$$

If $a = 0.3$ mm of precipitable water, $T = 275°$, $e = 10$ mb, $h = 38$ m. Simpson assumed further that each layer contains 0.06 gm of carbon dioxide. These figures for water vapor and carbon dioxide were chosen under the assumption that they represent the amount of water vapor and carbon dioxide in the stratosphere. Using Hettner's data except in the region from 9 to 12μ, where according to Fowle[1] but contrary to Hettner no absorption takes place, he divided the spectrum in three zones with respect to the absorption by such an atmospheric layer.

1. Practically complete absorption from 5.5 to 7μ and above 14μ.
2. Complete transparency from 8.5 to 11μ and below 4μ.
3. Incomplete absorption from 4 to 5.5μ, from 7 to 8.5μ, and from 11 to 14μ.

In the light of Weber's and Randall's measurements, which gave smaller absorption coefficients than those found by Hettner, it will be necessary to modify Simpson's assumption, in particular by choosing thicker atmospheric layers of larger water content in order to ensure complete absorption in the regions mentioned under (1) (see also page 103).

The absorption coefficient of liquid water[2] is so large that droplets of a size such as that of the droplets in clouds and fogs emit and absorb radiation practically like black bodies.

35. The Effect of the Line Structure of the Water-vapor Spectrum on the Atmospheric Emission and Absorption. Albrecht[3] has discussed the water-vapor spectrum in the long-wave region from the viewpoint of quantum theory and has pointed out that the absorption coefficients should depend on the pressure and the temperature of the atmosphere. Elsasser[4] has

[1] Fowle, F. E., *Smithsonian Misc. Coll.*, **68**, No. 8, 1917.

[2] Rubens, H., and Ladenberg, E., *Ver. deut. physik. Ges.*, **11**, 16, 1909.

[3] Albrecht, F., *Met. Z.*, **66**, 476, 1931.

[4] Elsasser, W. M., *Monthly Weather Rev.*, **65**, 323, 1937; **68**, 175, 1938.

studied this question in greater detail on the basis of the new measurements by Weber and Randall[1] and by Randall, Dennison, Ginsburg, and Weber.[2]

The energy distribution in a spectral line can be represented by a curve of the form shown in Fig. 20 with the frequency ν as abscissa. The center of the line is at ν_0. The same curve represents, of course, according to Kirchhoff's law (29.1), the intensity of the absorption coefficient due to the line. From Fig. 20, it can be seen that the absorption may overlap, especially the absorption of strong lines that are spaced not too widely apart. For practical computations, this effect has to be smoothed out so that the variation of absorption with the wave length can be represented by a reasonably regular curve. When the absorption of radiation in a finite wave-length interval is considered, Beer's law

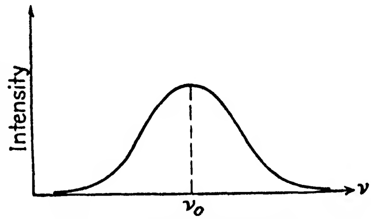

FIG. 20.—Energy distribution in a spectral line.

$$I_\lambda = I_{\lambda 0} e^{-k_\lambda u} \tag{34.11}$$

does not in general hold any more. The exponential function must be replaced by a more general transmission function

$$\tau_l(u) = \frac{I}{I_0} \tag{35.1}$$

where u is again the optical thickness of the medium. Elsasser[3] has shown that a good approximation for τ_l is given by

$$\tau_l(u) = 1 - \Phi\left(\sqrt{\frac{lu}{2}}\right) \tag{35.2}$$

where Φ is the error function

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx$$

[1] WEBER and RANDALL, *loc. cit.*

[2] RANDALL, H. M., DENNISON, D. M., GINSBURG, N., and WEBER, R. L., *Phys. Rev.*, **52**, 160, 1937.

[3] ELSASSER, W. M., *Phys. Rev.*, **54**, 126, 1938.

The quantity l, which measures the absorption due to the group of lines, may be called the "generalized absorption coefficient." Schnaidt has given a transmission function of a different form which is, however, in good numerical agreement with (35.2). In the region of weak absorption from 8 to 27μ, Beer's law (34.11) holds according to the theory[1] and according to the observations.[2]

On the basis of the observed absorption in water vapor, Elsasser[3] has computed generalized absorption coefficients l (or the ordinary absorption coefficient k) in the region from 8 to 27μ. The result of his computations is shown in Fig. 21, where the decadic logarithm of the absorption coefficient is plotted against the reciprocal of the wave length. The values of the wave lengths in microns have been added underneath. The region where the transmission function may be represented by an exponential is indicated by broken lines. The region around 15μ where a large part of the absorption due to carbon dioxide takes place[4] is indicated by vertical lines.

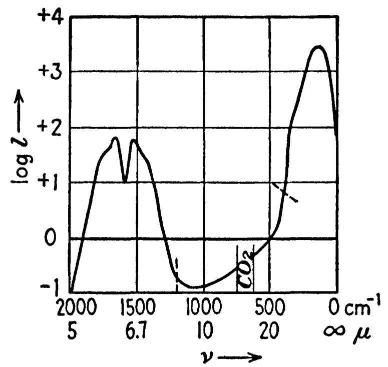

FIG. 21.—Atmospheric-absorption coefficients. (*After Elsasser.*)

The absorption curve shown in Fig. 21 may have to be subjected to corrections, especially around 6μ where[5] at present the data are not yet very reliable.

Elsasser has concluded from the theory that the absorption is proportional to the air pressure. According to observations of Falckenberg[6] which have been discussed by Schnaidt[7] it would appear, however, that the absorption is proportional rather to the

[1] ELSASSER, W. M., *Phys. Rev.*, **53**, 768, 1938.
[2] ADEL, A., *Astrophys. J.*, **87**, 497, 1938.
[3] ELSASSER, W. M., *Quart. J. Roy. Met. Soc.*, **66**, suppl., 41, 1940.
[4] CALLENDAR, G. S., *Quart. J. Roy. Met. Soc.*, **67**, 31, 1941.
[5] FOWLE, F. E., *Smithsonian Misc. Coll.*, **68**, No. 8, 1917.
[6] FALCKENBERG, G., *Met. Z.*, **53**, 172, 1936; **55**, 174, 1938.
[7] SCHNAIDT, F., *Gerl. Beitr. Geophys.*, **54**, 203, 1939.

square root of the pressure. The effect of the pressure can be taken into account as a correction to the optical path u.

There is also a slight temperature effect on the absorption, but it is so small that it may be neglected.

36. General Survey of the Terrestrial Heat Balance. Because the yearly mean temperature of the earth and its atmosphere remains constant apart from possible long periodic variations, the total amount of radiation received must be equal to the total amount of radiation emitted back into space. In a detailed study of the balance between the heat received and that emitted, each latitude should be considered separately. For a first orientation, however, it is useful to see how the different items of the terrestrial heat balance are distributed on the average.

Estimates of the average distribution of the items in the atmospheric heat balance have been given by various authors. Naturally, they differ somewhat, but the differences are not very important when only a first orientation is desired. The estimates of Baur and Philipps[1] will be given, with a modification suggested by Möller.[2]

According to (31.1) the total amount of solar radiation received per square centimeter and per day is $1.94 \times \frac{1440}{4} = 700$ cal at the outer limit of the atmosphere. Of this amount, 27 per cent penetrates directly to the earth's surface, and 16 per cent arrives as diffuse sky radiation, so that altogether 43 per cent reaches the ground. Fifteen per cent is absorbed by the atmosphere, including clouds. The rest, 42 per cent, is reflected back into space, 33 per cent by direct reflection on clouds and on the surface and 9 per cent by diffuse reflection. This value of the albedo, 0.42, differs somewhat from Aldrich's and from Danjon's values. A final value for the earth's mean albedo has not as yet been determined.

The distribution of the incoming radiation over the various items is shown on the left side of Fig. 22. The radiation received by the atmosphere and the earth is counted positive; the radiation given off by the atmosphere and the earth is counted negative.

[1] Baur, F., and Philipps, H., *Gerl. Beitr. Geophys.*, **45**, 82, 1935; **47**, 218, 1936.

[2] Möller, F., *Gerl. Beitr. Geophys.*, **47**, 215, 1936.

Figure 22 represents, strictly speaking, the yearly mean for the Northern Hemisphere only. But the conditions for the Southern Hemisphere have not yet been studied sufficiently.

While 42 per cent of the solar radiation is reflected directly back to space as short-wave radiation, the remaining 58 per cent is reemitted by the surface of the earth and the atmosphere in the form of long-wave radiation. The net outgoing radiation from the ground is 24 per cent. This represents the difference between the total radiation emitted by the ground, 120 per cent of the total solar radiation, and the radiation from the air

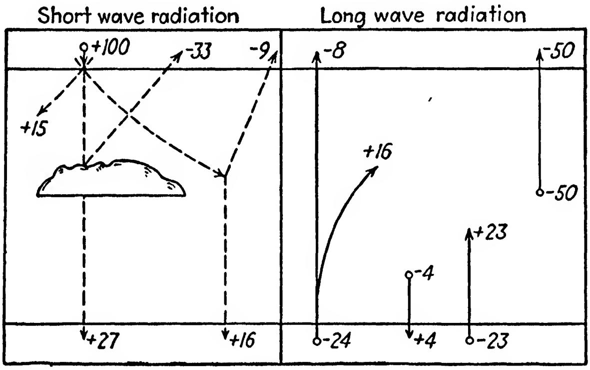

FIG. 22.—The heat balance on the Northern Hemisphere. Units are percentages of the incoming solar radiation. (*After Baur, Philipps, and Möller.*)

to the ground which is 96 per cent. Of this 24 per cent, 16 per cent is reabsorbed in the atmosphere and 8 per cent returns directly to space. The remaining 50 per cent is radiated back to space by the atmosphere. The distribution of the outgoing long-wave radiation over the various items in the terrestrial heat balance is represented on the right side of Fig. 22.

There is finally an internal transport of energy by turbulent mixing (Chap. XI) from the atmosphere to the surface, amounting to 4 per cent, and from the surface to the atmosphere by condensation. The latter was estimated to 23 per cent by Baur and Philipps. It takes place in the form of water vapor which carries along, crudely speaking, the latent heat of condensation and releases it when condensation begins at higher levels.

Including the heat transport by turbulence and by condensation a heat balance exists also for the earth's surface and the atmosphere separately.

The surface

receives		loses	
by direct radiation	27%	by radiation	24%
by diffuse radiation	16%	by condensation	23%
by turbulence	4%		
	47%		47%

The atmosphere

receives		loses	
by absorption of solar radiation	15%	by radiation	50%
by absorption of radiation from the earth	16%	by turbulence	4%
by condensation	23%		
	54%		54%

In this heat balance the 96 per cent that is radiated back from the air to the ground is again omitted, and only the outgoing radiation from the ground is inserted.

37. The Geographical Distribution of the Outgoing Radiation. The figures of the preceding section are mean values from which the incoming and outgoing radiation at different latitudes deviate considerably. The geographical distribution of the incoming radiation has been discussed in Secs. 31 and 32. The geographical variation of the outgoing radiation will now be considered.

Simpson[1] has developed a method for the computation of the outgoing radiation from the earth and the atmosphere based on his simplified scheme of the absorption spectrum of water vapor and carbon dioxide (page 96). He has assumed further that the stratosphere contains at least 0.3 mm of precipitable water and 0.06 gm of carbon dioxide. Thus, radiation between 5.5 and 7μ and above 14μ would be completely absorbed in the stratosphere. The radiation in these spectral regions that leaves the atmosphere must have originated in the stratosphere. Finally, Simpson has made use of the fact that the outward flux of radiation for a given wave length must lie between the fluxes from black bodies at the temperatures T_0 and T_1; these are the temperatures at the lower and upper limits of the gas layer, respectively.

Under these assumptions the outgoing radiation in the region of complete absorption described under (1) on page 96 is the black-body radiation emitted at the stratosphere temperature in

[1] Simpson, G. C., *Mem. Roy. Met. Soc.*, **3**, No. 21, 1928.

this part of the spectrum. The radiation in the transparent region (2) (page 96) is the black-body radiation at the temperature of the earth's surface for these wave lengths. Both amounts can easily be determined graphically by plotting the energy distribution of black bodies at the temperature of the surface and at the temperature of the stratosphere. In the intermediate region of partial absorption the radiation is intermediate between the black-body radiation at the temperature of the surface and at the temperature of the stratosphere. Simpson chooses the mean between these two values which can similarly be determined graphically. This assumption cannot cause a serious error, for a somewhat different choice would not seriously affect the value of the total outgoing radiation.

The intensity of the outgoing radiation is influenced by the cloudiness. When the sky is totally overcast, the radiation from the earth's surface has to be replaced by the radiation from the upper surface of the cloud which radiates like a black body. Simpson assumes that the temperature of the surface of the clouds is uniformly 261° abs at all latitudes and that the mean cloudiness is $5/10$ everywhere.

The results of Simpson's computations are summarized in the following table.

INCOMING AND OUTGOING RADIATION

Latitude, deg	Incoming radiation, cal/cm² min	Outgoing radiation, cal/cm² min, after		
		Simpson	Albrecht	Baur-Philipps
0	0.339	0.271	0.305	0.296
10	0.334	0.282	0.304	0.299
20	0.320	0.284	0.299	0.298
30	0.297	0.284	0.288	0.291
40	0.267	0.282	0.276	0.269
50	0.232	0.277	0.263	0.253
60	0.193	0.272	0.250	
70	0.160	0.260	0.236	
80	0.144	0.252	0.225	0.240
90	0.140	0.252	0.225	

The second column gives the effective incoming radiation from which the losses due to the albedo have been deducted. The

third column shows the outgoing radiation. The mean outgoing radiation was found to be 0.271 cal cm^{-2} min^{-1} which agrees well with the values for the average incoming radiation given on page 94. The outgoing radiation varies very little with the latitude. This is partly due to Simpson's assumption that the cloud temperature is independent of the latitude. Thus, with totally overcast skies the outgoing radiation would increase toward the pole, for the fraction emitted by the cloud surface remains constant while the radiation from the stratosphere increases poleward with increasing stratosphere temperature. At latitudes below 35° the incoming radiation is larger than the outgoing radiation. Therefore a heat transport toward the poles must take place if the yearly mean temperature of each latitude is to remain constant. This heat transport which brings the surplus radiative energy from the latitudes below 35° to the higher latitudes and here makes up for the deficit is effected by the general circulation and will be discussed in greater detail in Sec. 95.

In a later paper, Simpson[1] discusses the distribution of the outgoing radiation over the globe and its variation with the season. Again it appears that the outgoing radiation is very uniform in time and in space.

In view of the newer investigations of the infrared absorption spectrum of water vapor, Simpson's assumptions about the absorptive power of water vapor will have to be modified (page 96). Instead of a layer containing only 0.3 mm of precipitable water, layers having a higher water-vapor content will have to be considered; thus, the thickness of the layer would increase according to (34.2). Nevertheless, at least in the lower atmosphere, Simpson's method gives a first approximation to the outgoing radiation. But it appears that the stratosphere does not contain even 0.3 mm of precipitable water.[2] Therefore, the radiation in the spectral regions of complete absorption must originate, at least partly, in the upper troposphere.

Because the decrease of the water vapor with the height follows an exponential law the main part of the radiation from the uppermost layer which still contains the critical amount of precipitable water must originate in the lower part of this layer.

[1] Simpson, C. G., *Mem. Roy. Met. Soc.*, **3**, No. 23, 1929.

[2] Brunt, D., and Kapur, A. L., *Quart. J. Roy. Met. Soc.*, **64**, 510, 1938.

This was pointed out by Albrecht[1] according to whom the "emission layer" is completely in the troposphere. The temperature of Albrecht's emission layer is more uniform than the temperature of the stratosphere. Therefore, the variation of the outgoing radiation computed by Albrecht varies more with the latitude than Simpson's values, for now the effects of the varying surface temperature are not so much compensated by the opposite temperature gradient in the upper layer. Albrecht's results are shown in the fourth column of the table on page 102.

Finally, the investigation by Baur and Philipps,[2] referred to in the preceding section, must be mentioned here. Baur and Philipps regard the temperature and water-vapor content of the air as given, so that the black-body radiation for every wave length and the relation between the height and the optical mass u are known. In order to eliminate the difficulties arising out of the complicated variation of the absorption coefficient k_λ with the wave length λ, they divide the spectrum in three regions and assume a certain constant value for each region. In the table on page 102 the yearly means of the outgoing radiation after Baur and Philipps are included. The figures are set halfway between the latitudes, for they refer to the latitudinal belts 0 to 10°, etc., and the polar zone from 60 to 90°.

The differences between the three sets of figures for the outgoing terrestrial radiation are easily understandable, for they are all based on approximate calculations and on simplifying assumptions. However, it is, interesting to note that in all three instances the outgoing radiation becomes larger than the incoming between 30 and 40° latitude.

38. Computation of the Radiation Currents in the Atmosphere. Owing to the complexity of the water-vapor spectrum the numerical computation of the radiation currents passing upward and downward through the atmosphere is exceedingly laborious. This is especially true if the evaluation is to be carried out not for an average temperature and moisture distribution but for a number of actually observed cases. On the other hand, a knowledge of the upward- and downward-going radiation permits one to find the changes of the energy and therefore of the temperature in the atmosphere, due to radiation.

[1] Albrecht, F., *Met. Z.*, **48**, 57, 1931.

[2] Baur, F., and Philipps, H., *Gerl. Beitr. Geophys.*, **45**, 82, 1935.

In order to make the computation of the radiation currents in the atmosphere practicable, Mügge and Möller have constructed a chart[1] by means of which the currents can be determined graphically, once the temperature and moisture distribution in the vertical direction are known. More recently, Elsasser[2] has constructed a radiation chart based on the new theoretical and experimental investigations of the infrared spectrum of water vapor. For the details of the theory and the practical use of the chart the reader is referred to Elsasser's paper.

A study of typical temperature and humidity distributions obtained from aerological soundings on the North American continent with the aid of the radiation chart has shown that the cooling of the atmosphere due to radiation is of the order of 1°C per day in polar air masses and of the order of 2 to 3°C per day in equatorial air masses.[3] An empirical equation of the form

$$\Delta T = 1 + 2 \log w \tag{38.1}$$

where ΔT represents the cooling in degrees centigrade per day and w, the specific humidity in grams per kilogram of moist air, represents the relation between cooling and humidity quite well, although all that can be claimed is that the relation is a satisfactory approximation in the lower troposphere and in middle latitudes.

Clouds usually gain some heat on their base, for the radiation from the ground and the lower atmospheric layers is larger than the radiation from the cloud base. At the same time, they lose heat at their top because the cloud surface radiates like a black body while it receives only the selective atmospheric radiation from the upper layers. The cooling of clouds per day can be represented approximately by the expression

$$\Delta T = \frac{1000}{\Delta p} \tag{38.2}$$

where Δp is the thickness of the cloud expressed in millibars.

It appears clearly that the atmosphere is not heated anywhere by radiation. It seems that the rate of cooling decreases steadily

[1] Mügge, R., and Möller, F., *Z. Geophysik*, **8**, 53, 1932.
[2] Elsasser, W. M., *Quart. J. Roy. Met. Soc.*, **66**, suppl., 41, 1940.
[3] Elsasser, W. M., *Monthly Weather Rev.*, **68**, 185, 1940.

with the elevation above 2 km, at least up to 5 km; at this level, Elsasser's calculations end. This process would in the course of several days lead to an appreciable stabilization of the atmospheric stratification so that the radiative cooling may well be a major stabilizing factor in the atmosphere.

39. Nocturnal Radiation and the Cooling of the Surface Layers. The surface of the earth radiates like a black body so that the total radiation emitted can be computed by Stefan-Boltzmann's law (29.4). On the other hand, the ground receives radiation from the atmosphere and, in daytime, from the sun. The amount of radiation received in daytime is generally larger than the amount lost. At night when the short-wave radiation from the sun and the scattered sky radiation are absent, the radiation emitted by the ground is as a rule greater than the radiation received from the atmosphere; for the ground radiates like a black body throughout, whereas the atmosphere radiates like a black body only in certain spectral regions. The difference between the radiation from the ground upward and the radiation received from the atmosphere at the ground is called the *net outgoing* or *nocturnal radiation.*

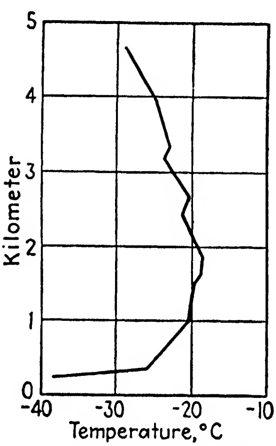

FIG. 23.—Vertical temperature distribution at Fort Smith, Northwest Territories, Jan. 28, 1937.

Wexler[1] has discussed how polar maritime air may be transformed into polar continental air by the cooling of the ground due to the net outgoing radiation. Polar maritime air shows the regular decrease of the temperature with the height, whereas polar continental air has as a rule a strong inversion in the lowest layers and then a more isothermal layer. Only at greater heights is the normal temperature gradient found. A typical example of the vertical temperature distribution in polar continental air is shown in Fig. 23.

Wexler computes with Simpson's method, but on the basis of the newer absorption measurements in the water-vapor spectrum by Weber and Randall, the radiation emitted by the

[1] WEXLER, H., *Monthly Weather Rev.*, **64**, 122, 1936.

air at a temperature T_A. Because the snow surface radiates like a black body, it can be found from the Stefan-Boltzmann equation (29.4) at which snow-surface temperature T_s the snow-surface radiation equals the atmospheric radiation. Within the range of temperatures that may occur the result can be expressed by the relation

$$T_A = 1.27T_S - 32 \tag{39.1}$$

When (39.1) is satisfied, the radiative energy loss of the snow surface is compensated by the atmospheric radiation. The direct solar radiation to the snow surface can be neglected even if the sun is above the horizon because its altitude at these high latitudes is very low. Furthermore, most of the solar radiation is reflected back to space by the snow surface.

Thus, if polar maritime air with an original surface temperature of 274° abs moves over a snow-covered continent, the snow-surface temperature must sink to 241° abs according to (39.1). But a state of radiative equilibrium between the snow surface and the air cannot persist for the air does not absorb all the radiation emitted by the snow surface. Therefore, it loses heat by radiation, and its temperature drops. At first a shallow ground inversion is formed. As the cooling spreads upward, an isothermal layer will develop from the top of the ground inversion to the height at which the temperature was the same as that of the air at the top of the inversion before the cooling. With the decrease of the air temperature the balance of radiation between atmosphere and snow surface is disturbed again and the snow-surface temperature must decrease further, a further cooling of the atmosphere being thus produced.

However, it must be added that this radiative formation of the ground inversion appears to be effective only at high latitudes where the surface temperature can fall much below the air temperature; for Elsasser[1] has shown that the radiative heat exchange between the ground and the atmosphere is concentrated in the lowest 50 m and is very small above this height, whereas the ground inversions often extend to 1 km and higher. The ordinary nocturnal inversion seems, therefore, almost exclusively of turbulent origin in so far as the transfer of heat between the ground and the air is concerned. It is of radiative origin only

[1] ELSASSER, M. W., *Monthly Weather Rev.*, **68**, 185, 1940.

in the sense that the heat loss of the ground itself is of a purely radiative nature.

When the nocturnal radiation R is found from observations, the radiation S from the atmosphere to the earth is given by

$$S = \sigma T^4 - R \tag{39.2}$$

where T is the surface temperature and S and R depend not only on the temperature but also on the water vapor. According to Ångström's[1] observations,

$$S = \sigma T^4(A - B\,10^{-\gamma e}) \tag{39.3}$$

Here e is the water-vapor pressure at the surface, and A, B, and γ are constants. When e is expressed in millibars, $A = 0.806$, $B = 0.236$, and $\gamma = 0.052$, but these constants cannot be regarded as final.

Brunt[2] has shown that the formula

$$S = \sigma T^4(a + b\sqrt{e}) \tag{39.4}$$

is also in good agreement with the observations. The constants a and b vary to a certain extent, their mean values[3] being $a = 0.44$ and $b = 0.080$ when the vapor pressure e is expressed in millibars.

Both Ångström's and Brunt's formulas must be considered as empirical formulas, although a theoretical justification of the first has been given by Ramanathan and Ramdas[4] and of the second by Pekeris.[5] Within the range of values of e that are likely to occur at the surface, both equations give about the same values of S with a proper choice of the constants. It is therefore not surprising that two formulas of apparently totally different form should have been deduced from the observations.

When $e = 0$, the formulas give a finite value for the atmospheric radiation. It is doubtful whether or not this extrapolation is permissible. Furthermore, the surface variations of the water vapor are only loosely connected with the changes of

[1] ÅNGSTRÖM, A., *Smithsonian Misc. Coll.*, **65**, No. 3, 1915.

[2] BRUNT, D., *Quart. J. Roy. Met. Soc.*, **58**, 389, 1932.

[3] BRUNT, D., "Dynamic Meteorology," 2d ed., p. 137, Cambridge University Press, London, 1939.

[4] RAMANATHAN, K. R., and RAMDAS, L. A., *Proc. Ind. Acad. Sci.*, **1**, 822, 1935.

[5] PEKERIS, C. L., *Astrophys. J.*, **79**, 441, 1934.

the water vapor at higher levels. Therefore, even with vanishing water-vapor pressure at the surface, there may still be an appreciable water-vapor radiation from higher altitudes. Moreover, there is also a certain amount of atmospheric radiation due to carbon dioxide.

When the sky is overcast, the nocturnal radiation from the ground is greatly reduced; for a cloud surface radiates like a black body. Therefore, the radiation from the atmosphere to the ground consists now not only of the water-vapor radiation but also of the radiation in the other spectral regions emitted at the temperature of the cloud base.

The radiation from the cloud base to the ground is greater the lower the cloud. The nocturnal radiation R from the ground should therefore increase with increasing cloud height. This appears to be borne out by an investigation of Ångström,[1] at least for low clouds below 1.5 km. According to Ångström,

$$R = 0.011 + 0.036h \tag{39.5}$$

where h is the cloud height in kilometers.

The amount of cloudiness affects also the intensity of R. Ångström and Asklöf[2] have from their observations derived the formula

$$R_m = (1 - km)R_0 \tag{39.6}$$

where R_m is the nocturnal radiation for the cloudiness m ($m = 0$, clear; $m = 10$, completely overcast) and k is a constant which for low clouds is 0.08.

The problem of the nocturnal radiation is important for the forecast of minimum temperatures. When the nocturnal radiation is strong, the cooling of the ground and of the adjacent layers of the air will also be considerable so that in spring, when the temperatures are not yet sufficiently above freezing, frost with subsequent damage to certain crops is likely to occur. However, nocturnal cooling depends on a number of other factors besides the radiation which is determined by the cloudiness and the water-vapor content of the atmosphere. (1) When a wind is blowing, potentially warmer air is brought down from

[1] Ångström, A., *Stat. Met.-Hydr. Anst. Stockholm, Upps.*, No. 8, 1936.

[2] Asklöf, S., *Geog. Ann.*, **2**, 253, 1920.

higher to lower levels by turbulent mixing (see Sec. 84). (2) The heat lost by the surface of the earth is partly replaced by conduction from below. The heat conducted from the lower layers of the ground to the surface of the earth depends on the heat capacity, the density, and the conductivity of the soil. These factors vary considerably for different soils and even for one and the same soil, depending on the water content and the amount of air space between the particles of soil.

A great number of empirical investigations into the problem of forecasting minimum temperatures have been undertaken. To these the reader is referred for details.[1]

40. The Differential Equations of Atmospheric Radiation. The differential equations of the atmospheric radiation currents may be derived here for the sake of completeness. An analytical solution of these equations under assumptions that agree reasonably well with the actual absorption conditions in the atmosphere appears quite impossible in view of the complexity of the water-vapor spectrum.

Consider a given horizontal level in the atmosphere above which the total water-vapor content or, more precisely, the total optical mass is u, the temperature at this level being T. The downward beam of radiation for a given wave length λ may be called B_λ and the upward beam A_λ. Both may be assumed parallel, E_λ may be the black-body radiation at this temperature and wave length which is given by Planck's law (29.2). The upward beam A_λ passing through an infinitesimal layer of optical mass du extending upward from u to $u - du$ suffers a change dA_λ. Owing to absorption in the layer du the intensity of A_λ decreases by $k_\lambda A_\lambda\, du$ according to (34.1), and owing to radiation of the layer the intensity increases by $k_\lambda E_\lambda\, du$ according to Kirchhoff's law (29.1). Because u *decreases* upward, the first term is to be reckoned positive and the second negative. Thus,

$$dA_\lambda = +k_\lambda A_\lambda\, du - k_\lambda E_\lambda\, du$$

Similarly, it is found for the downward beam that

$$dB_\lambda = -k_\lambda B_\lambda\, du + k_\lambda E_\lambda\, du$$

[1] A complete review, including the literature up to 1926, will be found in R. Geiger, "Das Klima der bodennahen Luftschichten," F. Vieweg & Sohn, Brunswick, Germany, 1927.

In this manner the equations of Schwarzschild[1] for radiative transfer are obtained.

$$\begin{aligned}\frac{dA_\lambda}{du} &= k_\lambda(A_\lambda - E_\lambda)\\ \frac{dB_\lambda}{du} &= -k_\lambda(B_\lambda - E_\lambda)\end{aligned} \tag{40.1}$$

When the total radiative energy received by a layer is equal to the total energy emitted upward and downward, *radiation equilibrium* is said to exist.

$$2\int_0^\infty k_\lambda E_\lambda\, d\lambda = \int_0^\infty k_\lambda(A_\lambda + B_\lambda)\, d\lambda \tag{40.2}$$

The direct practical application of these equations to atmospheric radiation problems is severely restricted by the complexity of the water-vapor spectrum, as was discussed in Secs. 34 and 35.

41. Radiation and the Stratosphere. The atmosphere, at least up to about 25 km—from this height direct observations can be obtained by means of sounding balloons—may be divided into a lower part, the *troposphere*, and an upper part, the *stratosphere.* The troposphere is characterized by a linear decrease of the temperature, the lapse rate being approximately 0.6°C/100 m. At a height of about 16 km in the tropics and about 10 km in the temperate latitudes, there is a sudden discontinuity in the temperature gradient; and at higher levels the temperature remains either constant or increases slightly, as shown in Fig. 24.[2]

The boundary between troposphere and stratosphere is called the *tropopause.* The tropopause and therefore the layer with decreasing temperature are higher in tropical than in temperate latitudes. Therefore, the temperature in the lower stratosphere is cooler over the equator than at the same altitude over higher latitudes. But the temperature inversion in the stratosphere is strongest in low latitudes so that at greater heights in the stratosphere the temperature becomes more uniform in a horizontal direction.

[1] Schwarzschild, K., *Nachr. Ges. Wiss. Göttingen, Math.-Nat. Kl.*, 1906, 41.

[2] Ramanathan, K. R., *Mem. Ind. Met. Dept.*, **25**, 5, 1930. *Nature*, **123**, 834, 1929.

Owing to its isothermal state the stratification of the stratosphere must be very stable, as follows from the considerations of Sec. 9. Under these conditions, vertical motions should be small and the temperature distribution in the stratosphere should be dominated by radiative processes, at least to a considerably greater extent than the tropospheric temperature distribution. Because the temperature, in the absence of other than radiative processes, can remain constant with time only when each layer emits as much as it absorbs, it was concluded

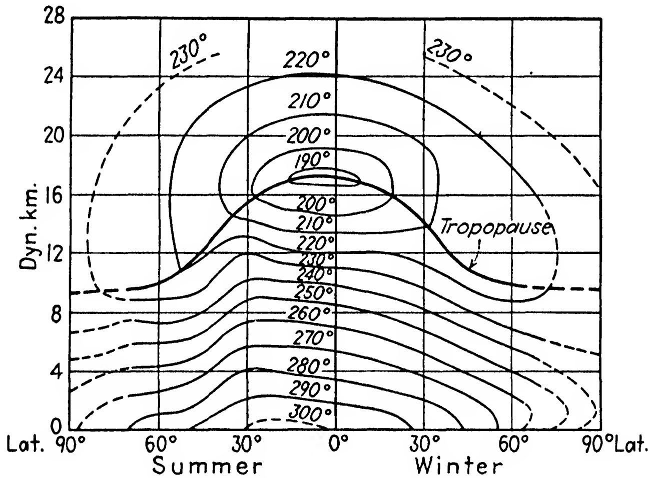

FIG. 24.—The temperature distribution over the Northern Hemisphere. (*After Ramanathan.*)

by a number of authors that the stratosphere must be in radiation equilibrium so that Eq. (40.2) is satisfied.

Humphreys[1] has developed a simple theory aiming at an explanation of the isothermal state and the temperature of the stratosphere, but Emden[2] showed that his treatment is open to objections. About the same time as Humphreys, Gold[3] and, somewhat later, Emden attempted explanations of the stratosphere. Both Gold's and Emden's papers were based on the assumption of gray radiation so that k_λ in Eqs. (40.1) for radiative transfer becomes a constant. For the difference

[1] HUMPHREYS, *Astrophys. J.*, **29**, 14, 1909, "Physics of the Air," p. 46.

[2] EMDEN, R., *Sitz.-Ber. Bayr. Akad. Wiss., Math.-Nat. Kl.*, 1913, 55.

[3] GOLD, E., *Proc. Roy. Soc.* (*London*), *A.*, **82**, 43, 1909.

between the "short" radiation from the sun and the long terrestrial radiation, allowance was made by assuming two different absorption coefficients, one for solar, the other for terrestrial radiation. Emden's results seemed for a while to establish a fairly satisfactory theory of the existence of the stratosphere, for he found that strong superadiabatic lapse rates would develop in the lower troposphere and isothermalcy in the stratosphere. But especially through the work of Simpson,[1] it became clear that the assumption of only two different absorption coefficients for the whole spectrum is too crude to yield satisfactory results and that the selective nature of atmospheric absorption has to be taken into account. The reader who desires more information about these attempts to explain the existence of the stratosphere as a consequence of radiation equilibrium is referred to an excellent critical summary by Pekeris.[2]

At present no satisfactory theory of the stratosphere exists. In fact, even the assumption that the atmosphere is in radiation equilibrium has to be rejected, as pointed out by Penndorf[3] and Brunt.[4] It appears that the role of such dynamic processes as advection of air of different temperature and even convective mixing is more important in the stratosphere than has been previously assumed.

A further barrier to the development of a theory of the temperature distribution in the stratosphere is the lack of knowledge of the amount of water vapor in the stratosphere, which is indispensable for a computation of the radiation currents.

Problem

8. Show that in the case of radiation equilibrium the difference between the total upward-going and downward-going radiation energy is independent of the optical mass.

[1] Simpson, G. C., *Mem. Roy. Met. Soc.*, **2**, No. 16, 1927.

[2] Pekeris, C. L., Mass. Inst. Techn., Met. course, Prof. notes, 5, 1932.

[3] Penndorf, R., *Veröffentlich. Geophys. Inst. Leipzig*, 2d *ser.*, **8**, 253, 1936.

[4] Brunt, D., *Quart. J. Roy. Met. Soc.*, **66**, suppl., 34.

CHAPTER VI

THE EQUATIONS OF MOTION OF THE ATMOSPHERE

42. Plane Motion in Polar Coordinates. To study the motion on a sphere such as the earth, polar coordinates are most appropriate. Before the equations of motion in three dimensions are developed, it will be useful to recapitulate briefly the representation of the motion in a plane by means of polar coordinates.

If the mass of the body whose motion is to be studied is m and if the force F acting on it has the components F_x and F_y the equations of motion are in Cartesian coordinates

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \end{aligned} \tag{42.1}$$

The dots indicate, here and in the following discussion, differentiation with respect to time, each dot indicating one differentiation.

If r is the radius vector of a point P and ψ its angle with the x-axis, it follows from Fig. 25 that

$$\begin{aligned} x &= r \cos \psi \\ y &= r \sin \psi \end{aligned} \tag{42.2}$$

It is seen from Fig. 25 that the radial component F_r and the tangential component F_ψ of the force F can be expressed by the Cartesian component with the aid of the following equations:

$$\begin{aligned} F_r &= F_x \cos \psi + F_y \sin \psi \\ F_\psi &= -F_x \sin \psi + F_y \cos \psi \end{aligned} \tag{42.3}$$

Analogous equations hold for the components of the acceleration of the particle. Thus, the radial acceleration is

$$\ddot{x} \cos \psi + \ddot{y} \sin \psi$$

and the tangential acceleration

$$-\ddot{x} \sin \psi + \ddot{y} \cos \psi$$

If Eqs. (42.2) are differentiated twice with respect to time and substituted in these expressions, it is found that the radial

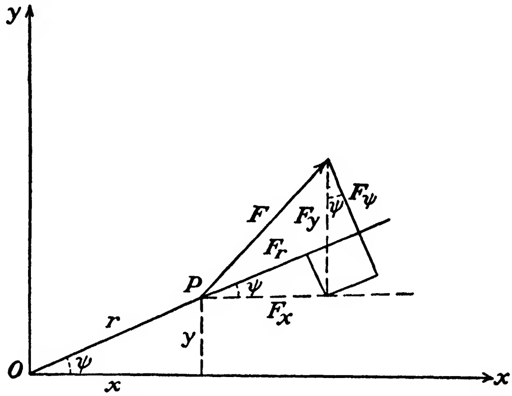

FIG. 25.—Plane motion in polar coordinates.

acceleration is

$$\ddot{r} - r\dot{\psi}^2 \tag{42.4}$$

and the tangential acceleration

$$r\ddot{\psi} + 2\dot{r}\dot{\psi} \tag{42.41}$$

The equations of motion in polar coordinates may now be written in the form

$$\begin{aligned} m(\ddot{r} - r\dot{\psi}^2) &= F_r \\ m(r\ddot{\psi} + 2\dot{r}\dot{\psi}) &= F_\psi \end{aligned} \tag{42.5}$$

The interpretation of the various terms is found in the textbooks on mechanics and need not be given here.

43. The Motion on a Rotating Globe. The equations of motion in a three-dimensional Cartesian coordinate system[1] are

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z \end{aligned}$$

where F_x, F_y, and F_z are the components of the external force. Because the earth may, at least to a very high degree of approxi-

[1] Only right-hand coordinate systems will be considered throughout this volume.

mation, be regarded as a sphere, polar coordinates may be introduced by the relations (Fig. 26)

$$
\begin{aligned}
x &= r \cos \varphi \cos \alpha \\
y &= r \cos \varphi \sin \alpha \\
z &= r \sin \varphi
\end{aligned}
\tag{43.1}
$$

r is the distance from the origin; φ the latitude, reckoned from the equator positive northward and negative southward; and α the longitude reckoned positive eastward from an arbitrary meridian.

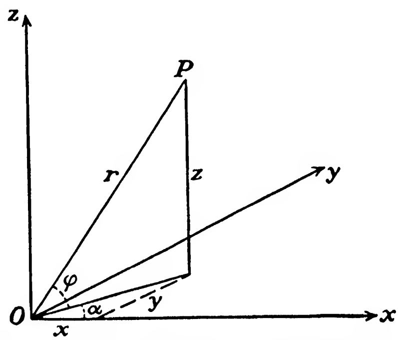

FIG. 26.—Transformation of Cartesian into spherical polar coordinates.

The equations of motion in spherical coordinates can be derived in a straightforward fashion, although with much labor, by differentiating Eq. (43.1) and substituting in the preceding equations. A shorter derivation is possible by considering the accelerations in the radial, longitudinal, and latitudinal directions and equating them to the components of the external force in these directions.

Consider the sphere in Fig. 27. O is the center of the sphere; AA' is its axis, which will later be identified with the axis of rotation but is arbitrary at present; and EE' is the equator. P is an arbitrary point on the sphere, $APEA'$ the meridian through P, and M the projection of P on the axis AA'. AGA' is the meridian from which the longitude is reckoned. The motion of P may be represented as the sum of

a. The motion in the meridian plane $APEA'$.

b. The rotation of the meridian plane due to variations of the longitude of P.

Owing to (*a*) the point has [see Eq. (42.4)] a radial acceleration $\ddot{r} - r\dot{\varphi}^2$ along OP and a tangential acceleration $r\ddot{\varphi} + 2\dot{r}\dot{\varphi}$, counted positive northward along the tangent to the meridian at P. Owing to (*b*), it has a radial acceleration $-r \cos \varphi \, \dot{\alpha}^2$ along MP, for $MP = r \cos \varphi$. A term of the form $\ddot{r}$ does not appear here, for the accelerations considered under (*b*) are due to the rotation of the meridian only. The tangential acceleration due to (*b*) is $r \cos \varphi \, \ddot{\alpha} + 2\dot{\alpha}(d/dt)(r \cos \varphi)$, parallel to the tangent at P to the latitudinal circle through P and directed eastward. The radial acceleration $-r \cos \varphi \, \dot{\alpha}^2$ may be resolved into two components, one along OP, $-r \cos^2 \varphi \, \dot{\alpha}^2$, the other one toward north, $r \cos \varphi \sin \varphi \, \dot{\alpha}^2$. Thus, the accelerations along the three rectangular directions are the following: Along OP,

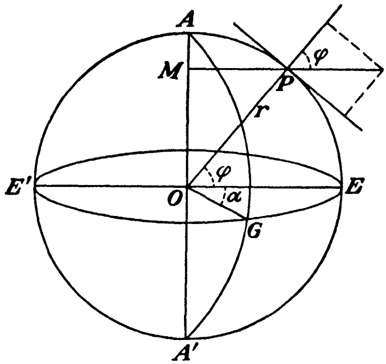

FIG. 27.—Acceleration in spherical coordinates.

$$\ddot{r} - r\dot{\varphi}^2 - r \cos^2 \varphi \, \dot{\alpha}^2 \tag{43.2}$$

Normal to OP, eastward,

$$r \cos \varphi \, \ddot{\alpha} + 2\dot{\alpha}(\dot{r} \cos \varphi - r \sin \varphi \, \dot{\varphi}) \tag{43.3}$$

Normal to OP, northward,

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r \cos \varphi \sin \varphi \, \dot{\alpha}^2 \tag{43.4}$$

These expressions for the accelerations hold for a coordinate system fixed in space. When a coordinate system rotating with the earth is used, r and φ remain unchanged provided that the axis AA' coincides with the axis of rotation. If λ is the longitude measured from a fixed point on the rotating earth,

$$\alpha = \lambda + \omega t$$

where ω is the angular velocity of the earth rotation. Upon substituting this relation in the expressions for the accelerations and equating to the components of the external force which

are F_λ in the longitudinal direction (toward E), F_φ in the meridional direction (toward N). and F_r in radial direction (upward), it follows that

$$r \cos \varphi \ddot{\lambda} + 2(\dot{\lambda} + \omega)(\dot{r} \cos \varphi - r \sin \varphi \dot{\varphi}) = \frac{F_\lambda}{m} \quad (43.51)$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r \cos \varphi \sin \varphi(\dot{\lambda} + \omega)^2 = \frac{F_\varphi}{m} \quad (43.52)$$

$$\ddot{r} - r\dot{\varphi}^2 - r \cos^2 \varphi(\dot{\lambda} + \omega)^2 = \frac{F_r}{m} \quad (43.53)$$

If a particle on the rotating earth is considered, which is originally at rest,

$$\dot{r} = \dot{\varphi} = \dot{\lambda} = 0$$

and only subjected to the acceleration of gravity g,

$$F_\lambda = 0 = F_\varphi \qquad \text{and} \qquad F_r = -mg$$

the preceding equations become

$$\begin{aligned} r \cos \varphi \ddot{\lambda} &= 0 \\ r\ddot{\varphi} + r \cos \varphi \sin \varphi \, \omega^2 &= 0 \\ \ddot{r} - r \cos^2 \varphi \, \omega^2 &= -g \end{aligned}$$

The two quantities $-r \cos \varphi \sin \varphi \, \omega^2$ and $r \cos^2 \varphi \, \omega^2$ are the meridional and vertical components of the centrifugal acceleration due to the earth's rotation (Fig. 27). A particle originally at rest on the rotating earth is thus subjected to the horizontal component $-r \cos \varphi \sin \varphi \, \omega^2$ of the centrifugal acceleration which is directed toward the equator. Because this force is acting on all masses on the earth, the figure of the earth is not strictly spherical but ellipsoidal with an equatorial radius that is slightly longer than the polar radius. Owing to the deviation of the earth from the true spherical form, the gravitation is not directed exactly to the center of the earth. In order to take the ellipsoidal form of the earth into account the coordinate system may be turned slightly around the tangent on the latitudinal circle at P (Fig. 27) so that the radius vector OP coincides with the true vertical. The angle between the true vertical and the radius vector OP is small, about $700''$ sin 2φ. In this new coordinate system the meridional component of the centrifugal acceleration is compensated by a horizontal component of the acceleration of

gravity. The vertical component of the acceleration of gravity, which is also slightly changed by the rotation of the coordinate system, is combined with the vertical component of the centrifugal acceleration of the earth's rotation. Only the sum of both can be determined by observation, and this sum is what is now denoted as the *acceleration of gravity.* In Eqs. (43.51), (43.52), and (43.53) the centrifugal terms may be omitted now, so that

$$r \cos \varphi\, \ddot{\lambda} + 2(\dot{\lambda} + \omega)(\dot{r} \cos \varphi - r \sin \varphi\, \dot{\varphi}) = \frac{F_\lambda}{m} \tag{43.61}$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r \cos \varphi \sin \varphi\, \dot{\lambda}(\dot{\lambda} + 2\omega) = \frac{F_\varphi}{m} \tag{43.62}$$

$$\ddot{r} - r\dot{\varphi}^2 - r \cos^2 \varphi\, \dot{\lambda}(\dot{\lambda} + 2\omega) = \frac{F_r}{m} \tag{43.63}$$

F_λ, F_φ, and F_r are the components of the external force in the new coordinate system. After the dynamic effects of the spheroidal figure of the earth have been taken into account, the earth may again be regarded as a sphere for most problems of dynamic meteorology.

44. The Conservation of Angular Momentum. Equation (43.61) can be written

$$\frac{1}{r \cos \varphi} \frac{d}{dt} [r^2(\dot{\lambda} + \omega) \cos^2 \varphi)] = \frac{F_\lambda}{m}$$

It follows, when the component of the external force in the west-east direction vanishes, that

$$r^2(\dot{\lambda} + \omega) \cos^2 \varphi = \text{const} \tag{44.1}$$

$r \cos \varphi$ is the distance from the axis of rotation, and $\dot{\lambda} + \omega$ the absolute angular velocity of the mass under consideration. Thus $r^2 \cos^2 \varphi(\dot{\lambda} + \omega)$ is the angular momentum, and (44.1) states the theorem of the conservation of angular momentum. If a mass is brought from a latitude φ_1 to a latitude φ_2 while r remains constant (the mass remaining at the same height),

$$\dot{\lambda}_2 + \omega = (\dot{\lambda}_1 + \omega) \frac{\cos^2 \varphi_1}{\cos^2 \varphi_2} \tag{44.2}$$

Because the linear velocity relative to the earth in the west-east

direction at φ_2,

$$u_2 = \dot{\lambda}_2 r \cos \varphi_2$$

and at φ_1

$$u_1 = \dot{\lambda}_1 r \cos \varphi_1$$

it follows that

$$u_2 = u_1 \frac{\cos \varphi_1}{\cos \varphi_2} + \omega r \frac{\cos^2 \varphi_1 - \cos^2 \varphi_2}{\cos \varphi_2} \tag{44.3}$$

Let us assume that at φ_1 the mass was at rest with respect to the earth, $u_1 = 0$. When the mass is brought from a lower to a higher latitude, $\cos \varphi_1 > \cos \varphi_2$ on either hemisphere, $u_2 > 0$. The mass acquires a velocity towards the east. By motion toward lower latitudes a westward velocity results. The effect is obviously due to the variation in the distance from the axis of rotation, $r \cos \varphi$, which must be compensated by a variation of $\dot{\lambda}$ according to (44.1).

The following table shows the velocities that result when a mass originally at rest is moved horizontally from a given latitude 10° toward the pole or the equator under conservation of its angular momentum:

Original latitude, deg	Displaced 10° toward pole, m/sec	Displaced 10° toward equator, m/sec
0	14	
10	42	14
20	71	41
30	99	66
40	126	88
50	152	105
60	181	118
70	232	123
80	∞	117
90	. . .	81

Such high velocities as would result from a meridional displacement, especially at higher latitudes, are rarely, if ever, observed in the atmosphere. It must therefore be concluded that large meridional displacements of air masses under con-

servation of angular momentum hardly occur in the earth's atmosphere, at least at higher latitudes.

From Eq. (44.1), it is also seen that a westward velocity results when a mass originally at rest is lifted and an eastward velocity when it is lowered. But the resulting velocities are quite small compared with the velocities arising from meridional displacements.

45. Introduction of a Cartesian Rectangular Coordinate System. In many problems of dynamic meteorology, it is permissible to neglect the curvature of the earth. It can then be assumed that the part of the earth on which the motion takes place coincides with a plane tangential to the earth. The z-axis may point vertically upward, the x-axis toward east, the y-axis toward north. The equations of motion in this coordinate system can be obtained from Eqs. (43.61) to (43.63) for spherical coordinates or by direct calculations.[1] The latter method will be used here even though it is somewhat lengthy, for it does not require any knowledge of the kinematics of a rigid body.

Let us assume instead of this Cartesian coordinate system another one x'', y'', z'' whose z''-axis coincides with the axis of the earth and whose origin is at the center of the earth. This system may not rotate with the earth. When the components of the external force are denoted by $F_{x''}$, $F_{y''}$, $F_{z''}$, the equations of motion are

$$\begin{aligned} \ddot{x}'' &= \frac{1}{m} F_{x''} \\ \ddot{y}'' &= \frac{1}{m} F_{y''} \\ \ddot{z}'' &= \frac{1}{m} F_{z''} \end{aligned} \tag{45.1}$$

Now, assume a second coordinate system x', y', z' whose z'-axis coincides with the z''-axis of the first system and whose origin is also at the center of the earth, so that the $x'y'$-plane coincides with the equatorial plane of the earth. This system may rotate with the earth. When x' coincides with x'' at the time $t = 0$, the angle between the two axes is ωt at a time t. The relation between the two systems is, according to Fig. 28 (the z''- and

[1] By the use of vector analysis a very short derivation of these equations is possible. See H. Solberg, *Geofys. Pub.*, **5**, No. 9, 7, 1928.

z'-axes are perpendicular to the plane of the paper):

$$\begin{aligned} x'' &= x' \cos \omega t - y' \sin \omega t \\ y'' &= x' \sin \omega t + y' \cos \omega t \\ z'' &= z' \end{aligned} \tag{45.2}$$

Analogous equations hold for the transformation of the components of the external force from one coordinate system into the other.

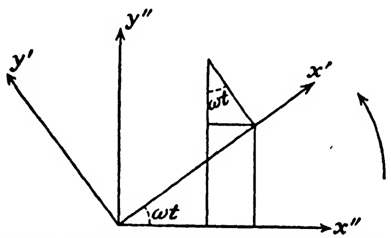

FIG. 28.—Transformation from a nonrotating into a rotating coordinate system.

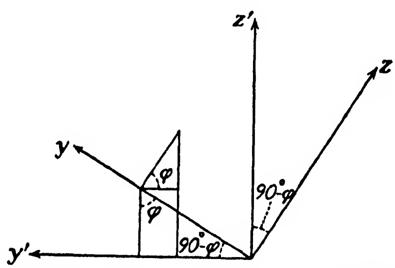

FIG. 29.—Transformation into a rotating coordinate system at an arbitrary latitude.

In order to transform the x', y', z' system into the x, y, z system mentioned at the beginning of the section, it has to be rotated around the x'-axis until the z'-axis points vertically upward at the latitude φ at which the motion is studied (Fig. 29).

In order to have the xy-plane tangential to the earth's surface at the latitude φ, a parallel translation of the x', y', z' system is also necessary, but this does not affect the dynamic equations. The xy-plane is then tangential to the earth. x may be directed toward east (perpendicular to and into the plane of the paper in Fig. 29) so that y points northward. The relation between the coordinate system x, y, z and x', y', z' is, according to Fig. 29,

given by

$$\begin{aligned} x' &= x \\ y' &= y \sin \varphi - z \cos \varphi \\ z' &= y \cos \varphi + z \sin \varphi \end{aligned} \tag{45.3}$$

From (45.2) and (45.3), it follows that

$$\begin{aligned} x'' &= x \cos \omega t - (y \sin \varphi - z \cos \varphi) \sin \omega t \\ y'' &= x \sin \omega t + (y \sin \varphi - z \cos \varphi) \cos \omega t \\ z'' &= y \cos \varphi + z \sin \varphi \end{aligned} \tag{45.4}$$

Again, analogous equations hold for the transformation of the components of the external force. Therefore

$$\begin{aligned} F_x &= F_{x''} \cos \omega t + F_{y''} \sin \omega t \\ F_y &= -F_{x''} \sin \varphi \sin \omega t + F_{y''} \sin \varphi \cos \omega t + F_{z''} \cos \varphi \\ F_z &= F_{x''} \cos \varphi \sin \omega t - F_{y''} \cos \varphi \cos \omega t + F_{z''} \sin \varphi \end{aligned} \tag{45.5}$$

With the aid of Eqs. (45.1) the last system of equations may be written

$$\begin{aligned} \ddot{x}'' \cos \omega t + \ddot{y}'' \sin \omega t &= \frac{1}{m} F_x \\ -\ddot{x}'' \sin \varphi \sin \omega t + \ddot{y}'' \sin \varphi \cos \omega t + \ddot{z}'' \cos \varphi &= \frac{1}{m} F_y \\ \ddot{x}'' \cos \varphi \sin \omega t - \ddot{y}'' \cos \varphi \cos \omega t + \ddot{z}'' \sin \varphi &= \frac{1}{m} F_z \end{aligned}$$

Upon differentiating (45.4) twice with respect to time (it should be noted that φ, which determines the position of the coordinate system x, y, z, is constant) these equations can be written, when the notations $\dot{x} = u$, $\dot{y} = v$, and $\dot{z} = w$ are introduced,

$$\begin{aligned} \frac{du}{dt} - \omega^2 x - 2\omega(v \sin \varphi - w \cos \varphi) &= \frac{1}{m} F_x \\ \frac{dv}{dt} - \omega^2(y \sin \varphi - z \cos \varphi) \sin \varphi + 2\omega \sin \varphi \, u &= \frac{1}{m} F_y \\ \frac{dw}{dt} + \omega^2(y \sin \varphi - z \cos \varphi) \cos \varphi - 2\omega \cos \varphi \, u &= \frac{1}{m} F_z \end{aligned} \tag{45.6}$$

The terms containing ω^2 are the components of the centrifugal acceleration, or rather their negative values. They need not be considered according to the discussion on page 118. The

equations of motion then take the form

$$\begin{aligned}
\frac{du}{dt} - 2\omega(v \sin\varphi - w \cos\varphi) &= \frac{1}{m} F_x \\
\frac{dv}{dt} + 2\omega \sin\varphi\, u &= \frac{1}{m} F_y \\
\frac{dw}{dt} - 2\omega \cos\varphi\, u &= \frac{1}{m} F_z
\end{aligned} \tag{45.7}$$

In the following discussions, the equations will be used mostly in this form.

Sometimes, however, it will be convenient to introduce a coordinate system $\bar{x}$, $\bar{y}$, $\bar{z}$ whose $\bar{x}$-axis makes an arbitrary angle β with the x-axis, *i.e.*, with the direction toward east, while the $\bar{z}$-axis remains perpendicular to the earth's surface. Analogous to (45.2) the equations of transformation between the old and the new coordinate system are

$$\begin{aligned}
x &= \bar{x} \cos\beta - \bar{y} \sin\beta \\
y &= \bar{x} \sin\beta + \bar{y} \cos\beta \\
z &= \bar{z}
\end{aligned}$$

Similar equations hold for the components of the velocity and of the acceleration, for β is independent of the time. Upon multiplying the first equation (45.7) by $\cos\beta$ and the second by $\sin\beta$ and adding, then multiplying the first equation by $-\sin\beta$ and the second by $\cos\beta$, adding, and substituting from the transformation equations for the components of the velocity and the force, it follows that

$$\begin{aligned}
\frac{d\bar{u}}{dt} - 2\omega \sin\varphi\, \bar{v} + 2\omega \cos\varphi \cos\beta\, \bar{w} &= \frac{1}{m} F_{\bar{x}} \\
\frac{d\bar{v}}{dt} + 2\omega \sin\varphi\, \bar{u} - 2\omega \cos\varphi \sin\beta\, \bar{w} &= \frac{1}{m} F_{\bar{y}} \\
\frac{d\bar{w}}{dt} - 2\omega \cos\varphi(\bar{u} \cos\beta - \bar{v} \sin\beta) &= \frac{1}{m} F_{\bar{z}}
\end{aligned} \tag{45.8}$$

46. The Coriolis, or Deflecting, Force of the Earth's Rotation. The terms containing the factor 2ω in the equations of the preceding section are the negative values of the components C_x, C_y, and C_z of the Coriolis force. Upon using the coordinate system x, y, z whose x-axis is directed toward the east,

$$\begin{aligned} C_x &= 2\omega(v \sin \varphi - w \cos \varphi) \\ C_y &= -2\omega \sin \varphi\, u \\ C_z &= 2\omega \cos \varphi\, u \end{aligned} \tag{46.1}$$

The direction cosines of the earth's axis are 0, cos φ, and sin φ in this coordinate system[1] (Fig. 30). Now,

$$0\, C_x + \cos \varphi\, C_y + \sin \varphi\, C_z = 0$$

Consequently, the sum of the products of the direction cosines of the Coriolis force into the respective direction cosines of the earth's axis is also zero. It follows that the Coriolis force is perpendicular to the earth's axis. Its absolute value is given by

$$\sqrt{C_x^2 + C_y^2 + C_z^2} = 2\omega\sqrt{(v \sin \varphi - w \cos \varphi)^2 + u^2} \tag{46.2}$$

To interpret this expression the coordinate system x', y', z' of Sec. 45 may be used, whose $x'y'$-plane is parallel to the equator. Analogous to (45.3),

$$\begin{aligned} u' &= u \\ v' &= v \sin \varphi - w \cos \varphi \end{aligned} \tag{46.21}$$

The Coriolis force is therefore equal to 2ω times the projection of the velocity into the plane of the equator.

Because $uC_x + vC_y + wC_z = 0$, the Coriolis force is perpendicular also to the velocity of the moving mass. Consequently, it can bring about changes only of the direction of the velocity but not of its amount. Therefore, it is also frequently called the *deflecting force* of the earth's rotation.

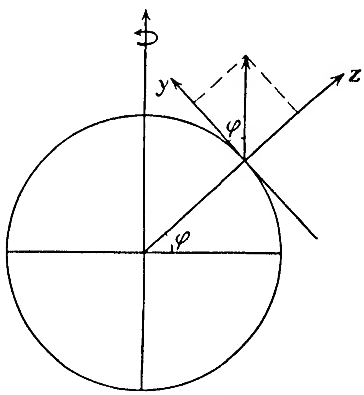

FIG. 30.—The direction cosines of the earth's axis.

When a purely horizontal motion is considered, $w = 0$, the first two of Eqs. (45.7) become

$$\begin{aligned} \frac{du}{dt} - 2\omega \sin \varphi\, v &= \frac{1}{m} F_x \\ \frac{dv}{dt} + 2\omega \sin \varphi\, u &= \frac{1}{m} F_y \end{aligned} \tag{46.3}$$

[1] In the coordinate system for which Eqs. (45.8) are derived the direction cosines are cos φ sin β, cos φ cos β, and sin φ.

The third equation may be omitted when horizontal motion is considered (see also page 145). The first two of Eqs. (45.8) for a coordinate system whose x-axis does not coincide with the direction toward east, assume the same form, for only the vertical component $\omega \sin \varphi$ of the angular velocity of the earth's rotation appears. The latter does not change with the transformation from the coordinate system x, y, z to x', y', z' as can be seen from Fig. 30. For horizontal motion,

$$C_x = 2\omega \sin \varphi\, v$$
$$C_y = -2\omega \sin \varphi\, u$$

From these expressions, it can again be seen that the Coriolis force in the two-dimensional case is perpendicular to the velocity, as was shown in three dimensions. Let OV in Fig. 31 represent the velocity. As drawn in Fig. 31, u and v are positive. Therefore, in the Northern Hemisphere where φ is positive, $C_x > 0$, $C_y < 0$, so that the Coriolis force falls in the direction OC, whereas in the Southern Hemisphere where φ is negative its direction is OC'. It follows that the deflective force acts to the right on the Northern Hemisphere, to the left on the Southern Hemisphere. The same result holds, of course, for three-dimensional motion, as can be shown by a somewhat involved discussion of the direction cosines. Because we shall not use this result in its general form, its derivation will not be given here.

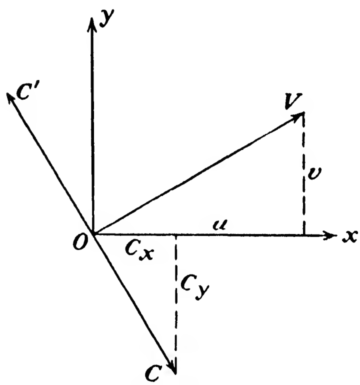

FIG. 31.—Direction of the Coriolis force in the horizontal.

At the equator, according to (46.1),

$$C_x = -2\omega w, \qquad C_y = 0, \qquad C_z = 2\omega u.$$

As long as purely horizontal motions and accelerations are considered, therefore, the Coriolis force does not appear when motions at the equator are studied. This is obvious; for the Coriolis force is parallel to the equatorial plane, and the equatorial plane is normal to the earth's surface at the equator.

47. The Hydrodynamic Equations. Because the atmosphere is a fluid medium, its motion follows the hydrodynamic equations. For a rigorous derivation of these equations the reader is referred to the textbooks on hydrodynamics. Here, the derivation of these equations will be outlined only briefly.

The effects of friction may be disregarded at present. They will be considered later in Chap. X. In a nonviscous fluid, each surface element is subjected to a pressure p, normal to the surface element. The pressure p is a continuous function of the coordinates x, y, z and the time t. Owing to the variations in space of p, a force is exerted on every fluid element. To find this force, consider a rectangular parallelepiped $dx\,dy\,dz$ whose edges are

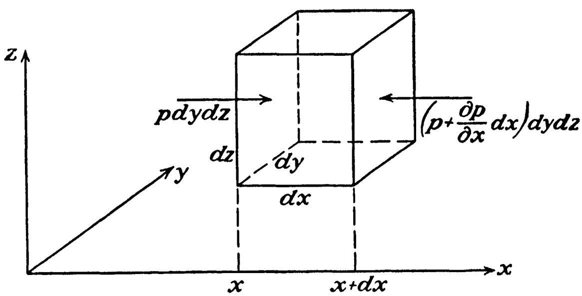

FIG. 32.—The computation of the pressure-gradient force.

parallel to the axes of the coordinate system (Fig. 32). Let the mean pressure on the face $dy\,dz$ with the abscissa x be p. Then the mean pressure on the face $dy\,dz$ with the abscissa $x + dx$ is $p + \dfrac{\partial p}{\partial x}\,dx$ and the forces exerted on both faces are $p\,dy\,dz$ and $\left(p + \dfrac{\partial p}{\partial x}\,dx\right) dy\,dz$, respectively. The latter force is directed toward decreasing x. The resultant of the two forces

$$-\frac{\partial p}{\partial x}\,dx\,dy\,dz$$

is the x-component of the force exerted on the volume element owing to the pressure variations. The force per unit mass is obtained by dividing by the mass of the volume element $\rho\,dx\,dy\,dz$. $\partial p/\partial x$ and ρ are mean values for the volume element; but when dx, dy, and dz tend to zero, they may be replaced by the values at the point x, y, z. The x-component of the force per unit mass,

$-\frac{1}{\rho}\frac{\partial p}{\partial x}$ has to be added to the expression for the x-component of the external force in (45.7) and (45.8). Similarly, to the y-component and z-component of the external force the expressions $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ and $-\frac{1}{\rho}\frac{\partial p}{\partial z}$ have to be added. These terms may be added explicitly to the equations of motion; for they will always be present, even if there is no external force acting. $\partial p/\partial x$ and $\partial p/\partial y$ are the horizontal components of the pressure gradient, $\partial p/\partial z$ the vertical component (see Sec. 6). In practical meteorology the negative values of these expressions are mostly referred to as the *components* of the pressure gradient. The expressions $-\frac{1}{\rho}\frac{\partial p}{\partial x}$, $-\frac{1}{\rho}\frac{\partial p}{\partial y}$, and $-\frac{1}{\rho}\frac{\partial p}{\partial z}$ are the components of the pressure-gradient force.

The terms du/dt, dv/dt, and dw/dt are the components of the acceleration of a moving fluid particle. They do not refer to a fixed point. The pressure gradient, on the other hand, is measured at a given fixed point. The accelerations have therefore to be resolved into quantities also referring to a fixed point in the fluid.[1] Because u, v, and w are functions of x, y, z, and t, it follows from the rules of partial differentiation that

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Further,

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w$$

Thus

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{47.1}$$

Corresponding equations hold for dv/dt and dw/dt. The partial derivatives of the velocity components with respect to the time are called the "local" derivatives because they measure the velocity changes at a given point. When the local derivatives

[1] It is also possible to transform the equations so that the pressure terms refer to a moving particle. If this method is used, the Lagrangian hydrodynamic equations are obtained; the equations derived here are the Eulerian equations.

vanish, the field is "steady." The other terms in (47.1) express the variation of the velocity of a moving particle due to its motion into regions of different velocity.

Thus we obtain the hydrodynamic *equations of motion* in a coordinate system which rotates with the earth and whose x-axis points eastward,

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\omega(v \sin \varphi - w \cos \varphi) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\omega \sin \varphi\, u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\omega \cos \varphi\, u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z
\end{aligned} \tag{47.2}$$

If it is more convenient to use a coordinate system whose x-axis includes an arbitrary angle with the x-direction, only the Coriolis

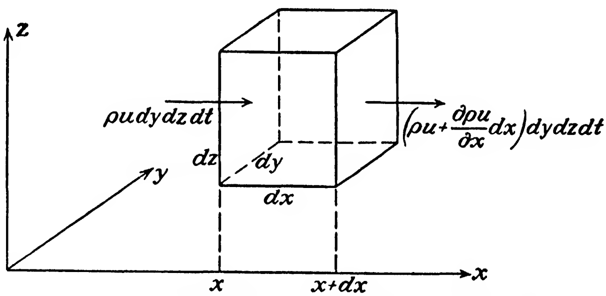

FIG. 33.—The derivation of the equation of continuity.

terms have to be changed as indicated by Eqs. (45.8). For the sake of brevity the accelerations will sometimes be written later in their undeveloped forms du/dt, dv/dt, and dw/dt. The operator d/dt refers to the differentiation of an individual particle with respect to the time. It may be called the "individual" time derivative in order to distinguish it from the local derivative $\partial/\partial t$.

To the equations of motion the condition of the continuity of mass has to be added. This condition states that in each volume of a fluid the net amount of mass entering through the (fictitious) boundary is equal to the increase of the mass enclosed by the boundary. To formulate this condition mathematically, consider Fig. 33. The fluid mass entering the parallelepiped

through the face $dy\,dz$ with the abscissa x in the time dt is $\rho u\,dy\,dz\,dt$ where ρu is a mean value for this area. Because $\left(\rho u + \frac{\partial \rho u}{\partial x}dx\right)$ is the corresponding value of the velocity for the face $dy\,dz$ with the abscissa $x + dx$, the fluid mass leaving through this area is $\left(\rho u + \frac{\partial \rho u}{\partial x}dx\right)dy\,dz\,dt$. The net gain of mass of the volume under consideration is $-\frac{\partial \rho u}{\partial x}dx\,dy\,dz\,dt$. Here, $\partial \rho u/\partial x$ is again a mean value which may be replaced by the value at the point x, y, z and at the time t when dx, dy, dz, and dt are sufficiently small. Similarly the net gains due to the flow in the y- and z-directions per unit time are $-\frac{\partial \rho v}{\partial y}dx\,dy\,dz\,dt$ and $-\frac{\partial \rho w}{\partial z}dx\,dy\,dz\,dt$, respectively. The sum of these three expressions represents the total net increase of the mass contained in this volume. This change in mass must cause a change of the density in the volume element from ρ at the time t to $\rho + \frac{\partial \rho}{\partial t}dt$ at the time $t + dt$. The mass increase in the time dt can therefore also be expressed by $\frac{\partial \rho}{\partial t}dt\,dx\,dy\,dz$. Upon equating both expressions for the mass increase and dividing by the volume $dx\,dy\,dz$ and the time dt the equation of continuity is obtained,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \qquad (47.3)$$

When the fluid under consideration is incompressible and homogeneous, $\rho =$ const, the three equations of motion (47.2) together with the equation of continuity (47.3) and appropriate boundary and initial conditions are sufficient to determine the motion because there are then four equations available to determine the four unknown variables u, v, w, and p. This is the case which is generally dealt with in hydrodynamics. For many meteorological problems, however, it is obviously not possible to regard the air as an incompressible and homogeneous fluid. Then ρ has to be added to the four other unknown variables, and a fifth equation has to be found in order to obtain a system of five equations from which to determine the five unknown quantities.

48. The Physical Equation. Piezotropy. The fifth equation in the system of hydrodynamic equations that must be available in order to make the number of equations equal to the number of unknown variables can be obtained from thermodynamic considerations. For ideal gases, for instance, we have the equation

$$p = \frac{R^*}{m} \rho T \tag{4.1}$$

between the pressure p, the density ρ, and the temperature T. However, if this equation is added to the four purely hydrodynamical equations, a sixth variable T appears so that a sixth equation is needed to make the system complete. The first law of thermodynamics may be chosen as this sixth equation,

$$dq = c_v\, dT + Ap\, d\left(\frac{1}{\rho}\right) \tag{8.1}$$

Here the amount of heat dq added to a particle has to be determined from other quantities, such as the heat received by radiation, by conduction, and by other processes.

Most problems investigated so far are of a much more limited scope, for it is assumed that the changes of the state of the fluid are polytropic (Sec. 8), in particular, adiabatic, or isothermal. In the case of polytropic changes

$$p\rho^{-\bar{\lambda}} = \text{const} \tag{8.51}$$

When this equation in which T does not appear is added to the three equations of motion and the equation of continuity, a complete system of five equations with five unknown variables is obtained, for a sixth variable does not enter. It should be noted that (8.51) depends on the coordinates, for the constant may change from particle to particle. An equation of the type (8.51) which contains only two of the three variables of state p, ρ, T is called a "piezotropic" equation after Bjerknes.[1]

The preceding example may be generalized. The equation of state for a particle may be given in the form

$$\rho = \rho(p, T, \cdots) \tag{48.1}$$

[1] Bjerknes, V., Bjerknes, J., Solberg, H., and Bergeron, T., "Physikalische Hydrodynamik," p. 84, Verlag Julius Springer, Berlin, 1933.

where the dots indicate that other variables of state besides p and T influence the density. In the case of atmospheric air, for instance, the water vapor would be such a variable. If, however, the density of the particle depends solely on one other variable of state, *e.g.*, on p, the fluid is called "piezotropic." The piezotropic equation will, in general, contain the coordinates x, y, and z. Thus, a piezotropic equation of state is of the form[1]

$$\rho = \rho(p; x, y, z) \tag{48.2}$$

of which (8.51) is a special case. When an equation of the type (48.2) can be added to the equations of motion and the equation of continuity, the number of equations is sufficient to determine the five unknown variables u, v, w, ρ, and p. The last equation will then be referred to as the *physical equation*. In an incompressible fluid the physical equation is, of course, superfluous.

A piezotropic fluid may be characterized by its *coefficient of piezotropy*. It is defined as the differential quotient $(d\rho/dp)_{ph}$. The subscript ph is added to indicate that this quantity refers to the *physical* changes of the state of the particle.

If the changes of state are isothermal, for instance, it follows from (4.1) that

$$\left(\frac{d\rho}{dp}\right)_{ph} = \frac{1}{(R^*/m)T}$$

Here T may change from place to place, for the assumption of piezotropy implies only that the changes of state of the individual particles are isothermal, but not the temperature distribution in space.

The physical equation (48.2) applies to a particle while the quantities appearing in (47.2) and (47.3) refer to a fixed point. Therefore, in Eq. (48.2), individual time derivatives may be formed,

$$\frac{d\rho}{dt} = \left(\frac{d\rho}{dp}\right)_{ph} \frac{dp}{dt}$$

If the individual time derivatives are developed in the same

[1] By the gas equation (4.1), ρ may, for instance, be replaced by T so that more generally, a piezotropic fluid is a fluid where the two others of the three variables p, ρ, and T can be found if only one variable is given.

manner as the accelerations in the preceding section,

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$
$$= \left(\frac{d\rho}{dp}\right)_{ph}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) \quad (48.21)$$

This last equation contains only quantities referring to a fixed point.

49. Barotropic and Baroclinic Stratification. When the spatial distribution of density and pressure in a fluid at a given moment is considered, the surfaces of equal density will, in general, not coincide with the surfaces of equal pressure. But if, for instance, the pressure and density distributions at a given time are functions of the altitude only, $p = p(z)$ and $\rho = \rho(z)$, z may be eliminated from these two equations, it being thus shown that $\rho = \rho(p)$. When an equation of this form holds for the distribution of density and pressure, the surfaces of equal pressure and density coincide. The fluid is then called *barotropic.* The general case where the surfaces of equal pressure are inclined to the surfaces of equal density is called the *baroclinic* case.

The properties barotropy and piezotropy should be clearly distinguished. The former refers to the density distribution in space, the latter to the dependence of the density of a particle on its pressure. In principle, the question whether the stratification of the atmosphere at a given time is baroclinic or barotropic can be decided from simultaneous observations at different points. To find out whether the atmosphere is piezotropic, observations of individual particles at different times are required.

In a barotropic fluid a coefficient of barotropy $(d\rho/dp)_g$ may be introduced. The subscript g indicates that this coefficient refers to the geometric distribution of ρ and p.

A fluid whose stratification is barotropic at a given moment does not, in general, remain barotropic. Consider, for instance, an atmosphere whose surfaces of equal pressure and equal density are horizontal and whose temperature decreases linearly with altitude, the lapse rate being less than adiabatic. Let the changes of state follow the adiabatic law so that the fluid is not only barotropic but also piezotropic. If a parcel of air is lifted in this atmosphere, it will arrive at its new position with a temperature lower than that of the surrounding air at the same level

and, therefore, with a higher density. Only if the original lapse rate of temperature had been adiabatic would the mass distribution have remained undisturbed and the barotropic stratification have been maintained.

A fluid whose equation of piezotropy is such that an original barotropic stratification is maintained is called *autobarotropic.* The simplest example for an autobarotropic fluid is an incompressible homogeneous atmosphere. Further examples are an isothermal atmosphere in which compression and expansion follow an isothermal law or an atmosphere with adiabatic lapse rate following an adiabatic law. The general condition for autobarotropy is that the piezotropic and the barotropic laws are identical so that the coefficients of piezotropy and barotropy become the same.

50. Streamlines. Divergence and Velocity Potential. To obtain a representation of the *instantaneous* state of motion of a fluid the streamlines are introduced. The *streamlines* have everywhere the same direction as the velocity so that their tangent indicates the direction of the velocity. The differential equations of the streamlines are therefore

$$dx:dy:dz = u:v:w \tag{50.1}$$

The streamlines coincide with the paths of the particles only if the fluid motion is steady. When the motion changes with time, the streamlines show the instantaneous distribution of motion throughout the fluid.

In the case of two-dimensional motion of an incompressible homogeneous fluid, the equation of continuity (47.3) assumes the simple form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{50.2}$$

It is satisfied by any function $\psi(x, y)$ provided that

$$u = -\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial x} \tag{50.3}$$

From the equation for the streamlines in the two-dimensional case,

$$dx:dy = u:v \tag{50.4}$$

it follows that along the streamlines

$$\psi(x, y) = \text{const} \tag{50.5}$$

The function $\psi(x, y)$ is therefore called the *stream function.*

From the derivation of the equation of continuity (47.3), it will be seen that the net amount of mass leaving the volume element $dx\,dy\,dz$ per unit time is

$$\left[\left(\frac{\partial \rho u}{\partial x}\right) + \left(\frac{\partial \rho v}{\partial y}\right) + \left(\frac{\partial \rho w}{\partial z}\right)\right] dx\,dy\,dz.$$

This expression measures the divergence of the flow of mass. More generally, the *divergence* of a vector with the components A_x, A_y, and A_z is

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{50.6}$$

When the divergence is negative at a point or in a region so that mass is accumulated there, it is sometimes called "convergence."

A function Φ is called the *velocity potential* of a given velocity distribution if

$$u = -\frac{\partial \Phi}{\partial x} \qquad v = -\frac{\partial \Phi}{\partial y} \qquad w = -\frac{\partial \Phi}{\partial z} \tag{50.7}$$

51. Circulation and Vorticity. The circulation C around a closed curve formed by fluid particles is the line integral of the velocity component tangential to the curve,

$$C = \oint V \cos \alpha\, ds \tag{51.1}$$

V is the velocity, ds the line element, and α the angle between V and ds. If u, v, and w are the components of the velocity V and dx, dy, and dz the components of ds, Eq. (51.1) may be written

$$C = \oint (u\,dx + v\,dy + w\,dz) \tag{51.2}$$

If the curve whose circulation is to be determined lies in the xy-plane,

$$C = \oint (u\,dx + v\,dy) \tag{51.21}$$

In Fig. 34, a curve in a horizontal plane is shown; P_1 and P_2 are two points on the curve a small distance ds apart, and O is the instantaneous center of rotation. Let $\frac{1}{2}\zeta$ be the angular velocity at P_1 with O as center. The contribution dC of the line element ds to the circulation is then, according to (51.1),

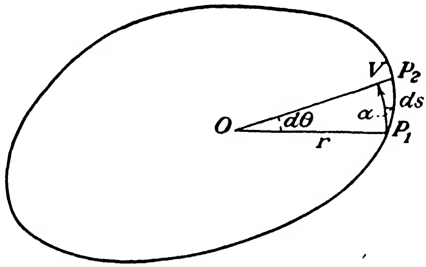

FIG. 34.—Circulation in a horizontal plane.

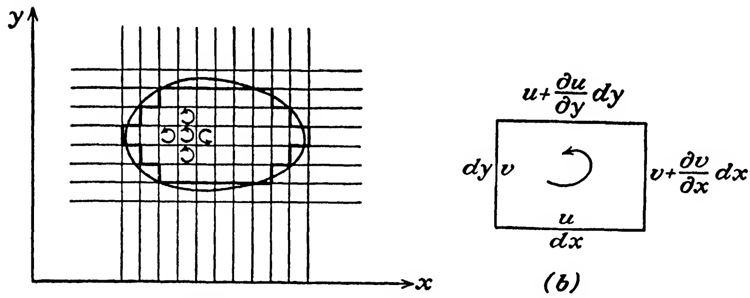

FIG. 35.—Circulation in rectangular Cartesian coordinates.

$r\frac{1}{2}\zeta \cos \alpha \, ds$ where α is the angle between the velocity and the line element at P_1. Because $\cos \alpha = \dfrac{r\,d\theta}{ds}$,

$$dC = r^2 \frac{1}{2} \zeta \, d\theta$$

and, by integration over the whole curve,

$$C = \oint r^2 \frac{1}{2} \zeta \, d\theta \tag{51.3}$$

r and $\frac{1}{2}\zeta$ are functions of θ. But when the curve is a circle with O as center whose radius is so small that ζ may be regarded equal

to the value at O everywhere in the region of the circle,

$$C = \pi r^2 \zeta \tag{51.31}$$

ζ is called the "vorticity" or the "curl" at O. It is equal to twice the angular velocity of the fluid at O.

Another expression for C may be obtained by dividing the area of the curve into small rectangles whose sides dx and dy are parallel to the x- and y-axes (Fig. 35*a*). The velocities tangential to the sides of an elementary rectangle are shown in Fig. 35*b*. It follows that the circulation around an elementary rectangle is given by

$$\begin{aligned} C_{dx\,dy} &= u\,dx + \left(v + \frac{\partial v}{\partial x}\,dx\right)dy - \left(u + \frac{\partial u}{\partial y}\,dy\right)dx - v\,dy \\ &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \end{aligned}$$

When the circulation for each rectangle in the contour is taken in the same (positive) direction and the circulations are added, the contribution from sides common to two rectangles cancel, for the summation is performed in opposite directions, as indicated in Fig. 35*a*. The sum represents the circulation around the broken curve approximating the original contour. Making the division of the area into rectangles sufficiently small,

$$C = \int\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \tag{51.4}$$

The surface integral is to be extended over the area enclosed by the curve around which the circulation is to be computed.

If (51.4) is applied to a circle whose radius r is so small that $(\partial v/\partial x) - (\partial u/\partial y)$ may be regarded as constant and equal to the value at the center O of the circle, it follows that

$$C = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \int\int dx\,dy \tag{51.41}$$

The double integral represents the area of the circle. Upon comparing this relation with (51.31), it is seen that the vorticity in the xy-plane (around the z-axis)

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{51.5}$$

Similarly, in the three-dimensional case, it is in the xz-plane,

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{51.51}$$

and in the yz-plane,

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \tag{51.52}$$

These expressions show how the vorticity can be computed when the field of the velocity is given.

From (51.4), (51.5), and (51.21), it is seen that in a plane

$$\oint (u\,dx + v\,dy) = \iint \zeta\,dx\,dy \tag{51.6}$$

This is Stokes's theorem for two-dimensional motion. Analogously, in three dimensions,

$$\oint (u\,dx + v\,dy + w\,dz) = \iint (\xi \cos nx + \eta \cos ny + \zeta \cos nz)\,dS \tag{51.7}$$

Here, $\cos nx$, $\cos ny$, $\cos nz$ are the direction cosines of the elements dS of the surface over which the integration is to be extended. The three-dimensional formula (51.7) will not be derived; for we shall not make use of it later, and the connection between vorticity and circulation is plain from (51.6). Both formulas show how a line integral is transformed into a surface integral.

When a velocity potential exists [see (50.7)], the vorticity vanishes[1] and the motion is called *irrotational*.

The reader should notice that the terms "rotational" and "irrotational motion" in the hydrodynamical sense do not always coincide in meaning with these terms as ordinarily used. Thus, a horizontal motion parallel to the x-axis that increases with the elevation z is rotational because $\partial u/\partial z \neq 0$. An example of irrotational circular motion, on the other hand, will be found in Prob. 11 at the end of this chapter.

52. The Circulation Theorems. The importance of the circulation C for dynamic meteorology is due to its close connection

[1] The circulation vanishes, also, provided that the potential is a single-valued function. But a discussion of such mathematical refinements may be omitted here.

with the vorticity. In order to find the growth and decay of vorticity the changes of the circulation with time may be studied. From (51.2), it follows that

$$\frac{dC}{dt} = \oint\left(\frac{du}{dt}\,dx + \frac{dv}{dt}\,dy + \frac{dw}{dt}\,dz\right) + \oint\left[ud\left(\frac{dx}{dt}\right) + vd\left(\frac{dy}{dt}\right) + wd\left(\frac{dz}{dt}\right)\right]$$

The second integral

$$\oint (u\,du + v\,dv + w\,dw) = \oint d\left(\frac{u^2 + v^2 + w^2}{2}\right) = 0$$

for the integration is extended over a closed curve. The remaining equation

$$\frac{dC}{dt} = \oint\left(\frac{du}{dt}\,dx + \frac{dv}{dt}\,dy + \frac{dw}{dt}\,dz\right) \tag{52.1}$$

states the relation between the variation of the circulation with time—frequently referred to as the "circulation acceleration"—and the acceleration of motion. It is known as *Kelvin's theorem.*

Upon substituting from the equations of motion (47.2) into (52.1), it follows that

$$\frac{dC}{dt} = -\oint\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right) + \oint (F_x\,dx + F_y\,dy + F_z\,dz) + \oint 2\omega[(v \sin\varphi - w\cos\varphi)\,dx - u\sin\varphi\,dy + u\cos\varphi\,dz]$$

The first integral can be written $-\oint (dp/\rho)$. The second integral vanishes if gravity is the only external force, for

$$\oint g\,dz = 0,$$

or, more generally, if the external force has a potential (which is a single-valued function).

In the third integral the coordinate system x', y', z' of Sec. 45 whose $x'y'$-plane is parallel to the equator may be introduced again with the aid of (45.3) and (46.21). Thus, this integral

becomes

$$2\omega \oint (v'\,dx' - u'\,dy')$$

Let the closed curve in Fig. 36 represent the equatorial projection of the curve along which the integration is to be performed. ds' with the components dx' and dy' is a line element of the projected curve. The variation per unit time of the area F enclosed by the pro ection of the original curve in the equatorial plane is due to the motion of all the line elements ds'. Because the velocity in the x'-direction is u', the change of the area due to the motion of ds' in the x'-direction is $u'\,dy'$ and similarly the change of the area due to the velocity in the y'-direction is $-v'\,dx'$. Thus

$$\frac{dF}{dt} = \oint (u'\,dy' - v'\,dx')$$

and

$$\frac{dC}{dt} = -\oint \frac{dp}{\rho} - 2\omega \frac{dF}{dt} \tag{52.2}$$

This is *V. Bjerknes's circulation theorem.* The integration is extended over a geometric curve. Therefore, when the *spatial* distribution of ρ can *always* be expressed by p, $\rho = \rho(p)$, *i.e.*, when the stratification of the fluid is autobarotropic, the change of the circulation is produced by the earth's rotation only. It follows that in a fluid with autobarotropic stratification in a nonrotating coordinate system the circulation and therefore the vorticity remain constant, especially zero, if they were zero to begin with. In this restricted form the theorem for an incompressible homogeneous fluid was originally derived by Helmholtz.

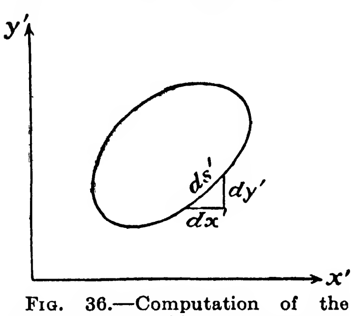

FIG. 36.—Computation of the effect of the earth's rotation on the circulation.

V. Bjerknes's theorem demonstrates clearly the importance of baroclinity for the dynamics of atmospheric motion. As long as the surfaces of equal pressure and density coincide, the circulation can change only owing to the effect of the earth's rota-

tion. In a baroclinic atmosphere, on the other hand, a change of the vorticity is brought about by the stratification of the atmosphere. The circulation theorem is a prognostic equation, for it gives the variation of the circulation with time in terms of the present distribution of pressure and density. Once a motion has begun, it is further modified by the effect of the earth's rotation as indicated by the second term on the right side of (52.2). It is interesting to note that the circulation integral $\oint \frac{dp}{\rho}$ has the same form as the integral appearing in (21.12) for the energy of a gas undergoing a cyclic process, although its physical interpretation is quite different.

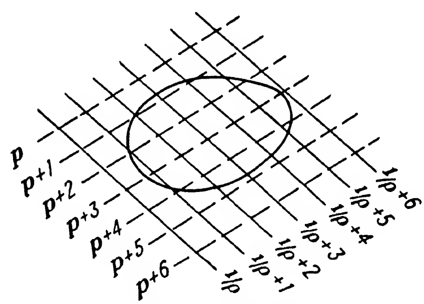

FIG. 37.—Solenoids.

To evaluate $\oint \frac{dp}{\rho}$, the surfaces of equal pressure, p = const (isobaric surfaces), and of equal specific volume, $1/\rho$ = const (isosteric surfaces), may be drawn at intervals corresponding to the system of units used (Fig. 37). These surfaces divide the space into tubes, the so-called "isobaric-isosteric solenoids" whose cross sections will be parallelograms. It can be shown that the value of $\oint \frac{dp}{\rho}$ is equal to the number of solenoids enclosed by the curve around which the circulation integral is to be taken.

Another form of $\oint \frac{dp}{\rho}$ can be obtained by means of Stokes's theorem (51.7). For the sake of brevity, only the case will be considered where the curve of integration lies in a plane. The following derivation is valid also for orientations of the plane

other than the horizontal. Because

$$\oint \frac{dp}{\rho} = \oint \left(\frac{1}{\rho}\frac{\partial p}{\partial x}\,dx + \frac{1}{\rho}\frac{\partial p}{\partial y}\,dy\right)$$

(51.6) may be applied. Here u is to be replaced by $\frac{1}{\rho}\frac{\partial p}{\partial x}$, v by $\frac{1}{\rho}\frac{\partial p}{\partial y}$. It follows that

$$\oint \frac{dp}{\rho} = \int\int \left[\frac{\partial(1/\rho)}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial(1/\rho)}{\partial y}\frac{\partial p}{\partial x}\right] dx\,dy \tag{52.21}$$

If α is the angle between the pressure gradient $\partial p/\partial n$ and the x-axis and β the angle between the gradient of specific volume $\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)$ and the x-axis,

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = \frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\cos\beta \qquad \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) = \frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\sin\beta$$

and

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial n}\cos\alpha \qquad \frac{\partial p}{\partial n} = \frac{\partial p}{\partial p}\sin\alpha$$

Thus

$$\oint \frac{dp}{\rho} = \int\int \frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial n}\sin(\alpha - \beta)\,dx\,dy$$

and

$$\frac{dC}{dt} = -\int\int \frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial n}\sin(\alpha - \beta)\,dx\,dy - 2\omega\frac{dF}{dt} \tag{52.3}$$

If $\alpha > \beta$, *i.e.*, if a negative rotation is necessary to bring the pressure gradient in the same direction as the gradient of the specific volume, $dC/dt < 0$, as shown in Fig. 38*a*. If $\alpha < \beta$, $dC/dt > 0$ (Fig. 38*b*). In both cases the circulation acceleration is from the pressure gradient to the gradient of specific volume.

For the practical computation of numerical values of dC/dt and for many theoretical discussions, it is convenient to introduce the temperature in $-\oint \frac{dp}{\rho}$. If the circulation in a vertical plane is to be computed, for instance, a convenient path of

integration would consist of two isobars p_0 and p_1 and two comparatively short vertical lines a and b connecting these isobars (Fig. 39).[1] In order to compute the intensity of the meridional component of the tropospheric circulation from the pole to the equator, one might choose the isobars 1000 mb and

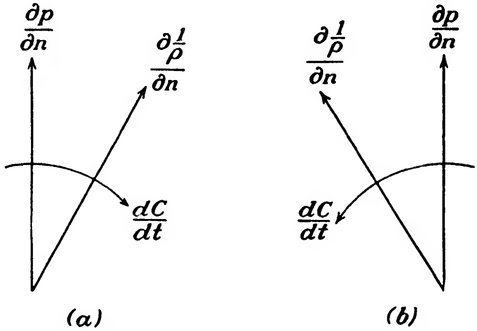

FIG. 38.—Direction of the circulation acceleration.

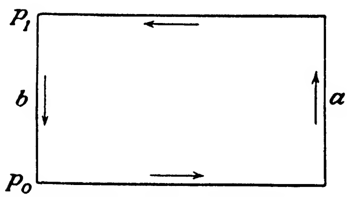

FIG. 39.—Computation of the circulation acceleration.

300 mb and the verticals connecting these isobars at the pole and at the equator (see also Sec. 94). From the equation of state (4.1), it follows that

$$-\oint \frac{dp}{\rho} = -R \oint T \frac{dp}{p}$$

Integration along the isobars gives zero. Along the curves a and b the mean temperatures $\overline{T_a}$ and $\overline{T_b}$, defined in analogy with (7.2), may be introduced so that

$$\frac{dC}{dT} = R(\overline{T_a} - \overline{T_b}) \ln \frac{p_0}{p_1} - 2\omega \frac{dF}{dt} \tag{52.4}$$

[1] BJERKNES, V., and collaborators, "Physikalische Hydrodynamik," p. 144, Verlag Julius Springer, Berlin, 1933.

If $\overline{T_a} > \overline{T_b}$, the circulation acceleration is in the direction indicated by the arrows in Fig. 39 so that the circulation is toward lower pressure (upward) on the warmer side, toward higher pressure (downward) on the colder side, and from the colder to the warmer side along the surface of higher pressure, and back from the warmer to the colder side along the surface of lower pressure.

The circulation theorem of V. Bjerknes can also be used to explain such small circulations as land and sea breezes or mountain and valley winds which may be mentioned briefly. In both cases the returning currents at upper levels appear to be less clearly established than the lower currents, but the circulation must nevertheless be closed in some form to satisfy the continuity condition. In the case of land and sea breezes, the air over the land is more strongly heated in daytime and cooled more strongly at night, for the water surface is kept at a more even temperature by mixing with the lower layers. If column a in Fig. 39 represents the air over land during the day and over water during the night, the origin of land and sea breezes is easily deduced.

Similarly, the air temperature near a mountain slope or in a V-shaped valley increases more in daytime and decreases more at night, owing to radiation, than the air at the same pressure level away from the mountain or outside the valley. Thus, there arises in daytime a wind blowing up the slope of the mountain or of the valley, and at night a wind blowing downward.[1]

Problems

9. Find the velocity that a particle acquires owing to the conservation of angular momentum if it is lifted or lowered.

10. Which form must the physical equation have in an autobarotropic atmosphere if the vertical lapse rate of temperature is constant and if the temperature is the same everywhere in the horizontal?

11. Show that a horizontal fluid motion along concentric circles is irrotational except at the center if the velocity is inversely proportional to the distance from the center.

12. Express the circulation integral $\oint \frac{dp}{\rho}$

a. By the gradients of pressure and potential temperature.
b. By the gradients of temperature and potential temperature.

[1] Wagner, A., *Met. Z.*, **49**, 329, 1932.

CHAPTER VII

SIMPLE ATMOSPHERIC MOTIONS

53. The Geostrophic Wind. When the air moves horizontally without change of the velocity and of the direction of motion and when the only external force is gravity, $F_x = F_y = 0$ and $F_z = -g$.[1] Equations (47.2) become

$$-2\omega \sin \varphi \, v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{53.11}$$

$$2\omega \sin \varphi \, u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{53.12}$$

$$-2\omega \cos \varphi \, u + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{53.13}$$

where the x-axis points toward east, the y-axis toward north, and the z-axis vertically upward. The equation of continuity (47.3) is satisfied when $\partial\rho/\partial t = 0$.

The coordinate system may be rotated around the z-axis so that the new x-direction coincides with the (positive) pressure gradient and makes an angle β with the east direction. The first two equations then become, according to (45.8),

$$-2\omega \sin \varphi \, v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{53.21}$$

$$u = 0 \tag{53.22}$$

and the third equation becomes, according to (45.8),

$$+2\omega \cos \varphi \, v \sin \beta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{53.23}$$

Here the expression on the left-hand side is of the order 10^{-3} m/sec^2 when v is of the order 10^1 m/sec and g is of the order 10^1 m/sec^2. Thus, Eq. (53.23) is a very close approximation to the static equation (6.1). With such a motion as that being

[1] Note that now forces per unit of mass are considered as follows from the expression for the pressure-gradient force.

considered now, static equilibrium may be assumed to prevail in the vertical direction.

The motion under discussion is referred to as the *geostrophic wind.* Equation (53.21) shows that under the conditions assumed equilibrium exists between the Coriolis force and the pressure-gradient force. Because the geostrophic wind is independent of space and time, the isobars plotted on a horizontal map must be straight lines in the geostrophic case.

Equation (53.21) gives the velocity of the geostrophic wind when the pressure gradient is known. It shows furthermore that the geostrophic wind is perpendicular to the pressure

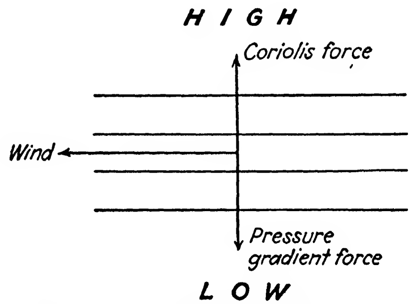

FIG. 40.—Geostrophic balance in the Northern Hemisphere.

gradient and therefore parallel to the isobars and in such a direction that in the Northern Hemisphere ($\varphi > 0$) the high pressure is to the right when one faces in the direction of the wind (Fig. 40). For instance, when $\partial p/\partial x > 0$ the higher pressure is toward positive x, while $v > 0$, so that wind blows in the positive y-direction.

The direction and velocity of the geostrophic wind can also be obtained directly by postulating that equilibrium should exist between the Coriolis force and the pressure-gradient force (Fig. 40). The latter is perpendicular to the isobars toward lower pressure. The Coriolis force must therefore also be perpendicular to the isobars and toward higher pressure. Because, in the Northern Hemisphere, the Coriolis force acts toward the right of the velocity and perpendicular to the wind velocity, the wind must be parallel to the isobars and in such a direction that the high pressure is to the right when one is facing in the direction of the wind. This rule is called *Buys Ballot's law.* In the Southern Hemisphere the wind blows, of course, in the

opposite direction. The condition that Coriolis force and pressure-gradient force must be numerically equal in order to balance leads again to (53.21).

54. The Inclination of Isobaric Surfaces. Instead of drawing the lines of equal pressure, the isobars, at a given level, *e.g.*, at sea level, it is sometimes more convenient to draw the height or dynamic height above sea level of an isobaric surface of a given pressure. Such a representation is called the "topography" or "dynamic topography" of the isobaric surface. If the wind is geostrophic, to a sufficient degree of approximation,

$$-2\omega \sin \varphi \, v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (53.21)$$

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (53.23)$$

when the x-axis is again in the direction of the (positive) pressure gradient. Upon multiplying the first of the equations by dx and the second by dz, it follows that

$$\frac{dp}{\rho} = +2\omega \sin \varphi \, v \, dx - g \, dz$$

Because the pressure on an isobaric surface is constant, the differential equation of the isobaric surface is given by

$$\frac{dz}{dx} = \frac{2\omega \sin \varphi}{g} v \qquad (54.1)$$

or when the dynamic height D is introduced according to (1.4),

$$\frac{dD}{dx} = \frac{2\omega \sin \varphi \, v}{10} \qquad (54.2)$$

According to these equations the geostrophic wind velocity is proportional to the inclination of the isobaric surfaces. The geostrophic wind velocity can therefore also be obtained from a map of the topography of an isobaric surface. In this case the density does not enter in the relation between gradient and wind.

Because $v > 0$ for $dz/dx > 0$ when $\varphi > 0$, it follows that the geostrophic wind blows in such a direction that in the Northern Hemisphere the regions of higher elevation of the isobaric sur-

face are to the right of the direction in which the wind blows (Buys Ballot's law).

The inclination of the isobaric surfaces is small. At a latitude of 43° an isobaric surface ascends 1 m in 10 km when the geostrophic wind velocity is 10 m/sec.

55. Horizontal Temperature Gradients and Geostrophic Motions. The equations for the geostrophic wind,

$$-2\omega \sin \varphi \, v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{53.11}$$

$$2\omega \sin \varphi \, u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{53.12}$$

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{53.13}$$

show that owing to vertical variations of the density the wind velocity and direction may be functions of the vertical coordinate even though they are constant in the horizontal. A comparison with Eq. (45.8) shows that (53.11) to (53.13) hold for any orientation of the x- and y-axes when $w = 0$, for the Coriolis term in the third equation can be neglected in comparison with g.

Eliminating the density by means of the gas equation

$$\rho = \frac{p}{RT} \tag{4.1}$$

from (53.11) to (53.13), it follows that

$$-2\omega \sin \varphi \frac{v}{T} = -R\frac{\partial \ln p}{\partial x} \tag{55.11}$$

$$2\omega \sin \varphi \frac{u}{T} = -R\frac{\partial \ln p}{\partial y} \tag{55.12}$$

$$\frac{g}{T} = -R\frac{\partial \ln p}{\partial z} \tag{55.13}$$

Differentiating with respect to z and substituting from the third into the other two equations,

$$-2\omega \sin \varphi \frac{\partial}{\partial z}\left(\frac{v}{T}\right) = \frac{\partial}{\partial x}\left(\frac{g}{T}\right) \tag{55.21}$$

$$2\omega \sin \varphi \frac{\partial}{\partial z}\left(\frac{u}{T}\right) = \frac{\partial}{\partial y}\left(\frac{g}{T}\right) \tag{55.22}$$

Integration between the levels z_0 and z shows that

$$\frac{u}{T} = \frac{u_0}{T_0} - \frac{g}{2\omega \sin \varphi} \int_{z_0}^{z} \frac{1}{T^2} \frac{\partial T}{\partial y} dz \qquad (55.31)$$

$$\frac{v}{T} = \frac{v_0}{T_0} + \frac{g}{2\omega \sin \varphi} \int_{z_0}^{z} \frac{1}{T^2} \frac{\partial T}{\partial x} dz \qquad (55.32)$$

Thus the geostrophic wind at a given level may be regarded as consisting of the surface wind changed in the ratio T/T_0 and a term depending on the horizontal temperature gradient. The latter is called the "thermal wind."[1] It should be clearly understood that this thermal wind is due only to the variation of the pressure-gradient force with the altitude. The wind was assumed to be geostrophic and must therefore reflect the variations of the pressure-gradient force.

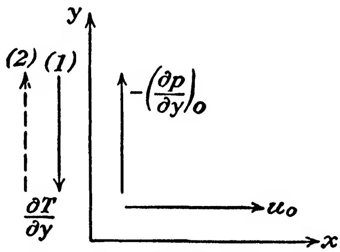

FIG. 41.—Velocity change of the geostrophic wind due to a horizontal temperature gradient.

When the pressure gradient is in the direction equal or opposite to the temperature gradient, the x-axis may be chosen parallel to u_0 (Fig. 41). The (positive) surface-pressure gradient is then directed in the negative y-direction (Sec. 53, Buys Ballot's law); and the temperature gradient is, according to the assumption, parallel to the y-axis. From (55.31), it follows

1. That the geostrophic wind increases with the altitude when the lower pressure coincides with the lower temperature.
2. That the geostrophic wind decreases with the altitude when the higher pressure coincides with the lower temperature.[2]

The same rules hold in the Southern Hemisphere.

Because the actual wind is often not very different from the geostrophic wind, the observed increase of the generally westerly currents of temperate latitudes upward throughout the troposphere can be explained as an effect of the temperature decrease toward the poles in the troposphere.

[1] BRUNT, D., "Physical and Dynamical Meteorology," 2d ed., p. 196, Cambridge University Press, London, 1939.

[2] MARGULES, M., *Met. Z.*, Hann-vol., 243, 1906.

When the surface-pressure gradient is normal to the temperature gradient, u_0 may again be directed along the positive x-axis and the pressure gradient along the negative y-axis. The temperature gradient is then, according to the assumption, directed along the positive or negative x-axis (Fig. 42). From (55.31), it follows that u remains constant apart from the effect of T/T_0, v is zero at z_0 and becomes positive or negative at higher levels. The following rules may be formulated.

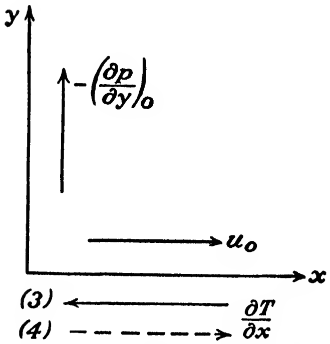

FIG. 42.—Turning of the geostrophic wind due to a horizontal temperature gradient.

3. When the geostrophic wind blows toward lower temperature, it turns to the right (veers).

4. When the geostrophic wind blows toward higher temperature, it turns to the left (backs).

In the Southern Hemisphere the wind veers when blowing toward the higher temperature and backs when blowing toward the lower temperature.

These four rules can easily be derived in a qualitative fashion when it is remembered that the pressure-gradient force decreases more slowly in warmer than in colder air. The rules are useful for forecasting the upper winds at times when upper-air observations are not available.

56. Steady Motion along Circular Isobars. In general the isobars are not straight lines but curved. In order to study the relation between pressure field and wind field, in this case, it is convenient to introduce horizontal polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \tag{56.1}$$

It will be assumed that the motion is steady, so that the local derivatives $\partial/\partial t$ vanish, and furthermore that the motion is purely horizontal. The equations of motion (47.2) become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\omega\sin\varphi\, v = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\omega\sin\varphi\, u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

To transform these into polar coordinates, it should be noted that, by differentiation of (56.1) with respect to time,

$$u = v_r \cos\theta - v_\theta \sin\theta$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$

Here

$$v_r = \frac{dr}{dt}, \text{ the radial velocity}$$
$$v_\theta = r\frac{d\theta}{dt}, \text{ the tangential velocity}$$

Further

$$\begin{aligned}\frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\\ \frac{\partial}{r\,\partial\theta} &= -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\end{aligned} \tag{56.2}$$

and

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = v_r\frac{\partial}{\partial r} + v_\theta\frac{\partial}{r\,\partial\theta} \tag{56.3}$$

Therefore, the equations of horizontal steady motion in polar coordinates are

$$\begin{aligned}v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{\partial v_r}{r\,\partial\theta} - \frac{v_\theta^2}{r} - 2\omega\sin\varphi\, v_\theta &= -\frac{1}{\rho}\frac{\partial p}{\partial r}\\ v_r\frac{\partial v_\theta}{\partial r} + v_\theta\frac{\partial v_\theta}{r\,\partial\theta} + \frac{v_\theta v_r}{r} + 2\omega\sin\varphi\, v_r &= -\frac{1}{\rho}\frac{\partial p}{r\,\partial\theta}\end{aligned} \tag{56.4}$$

It may now be assumed further that the isobars are concentric circles around the origin of the coordinate system so that $\partial p/\partial\theta = 0$. The second equation (56.4) can then be satisfied by assuming $v_r = 0$ and $\partial v_\theta/\partial\theta = 0$. Such a motion satisfies the condition of continuity. From the first equation, it follows that

$$+\frac{v_\theta^2}{r} + 2\omega\sin\varphi\, v_\theta = +\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{56.5}$$

which states that the motion is parallel to the isobars and that centrifugal, Coriolis, and pressure-gradient forces balance. This type of motion is called *gradient wind.* The geostrophic wind is obviously a special case of the gradient wind when the radius

of the isobars is infinite, *i.e.*, when the isobars are straight lines. It follows from the above assumption that the gradient-wind relation holds strictly only for circular isobars and circular streamlines coinciding with the isobars. But, in the case of other curved isobars, it gives at least a better approximation to reality than the geostrophic wind relation.

The gradient-wind equation (56.5) can be written down immediately as the condition that the centrifugal, Coriolis, and pressure-gradient forces balance, as shown in Fig. 43 for centers of low- and high-pressure distribution in the Northern Hemisphere.

In the case of a center of low pressure (Fig. 43*a*) the wind

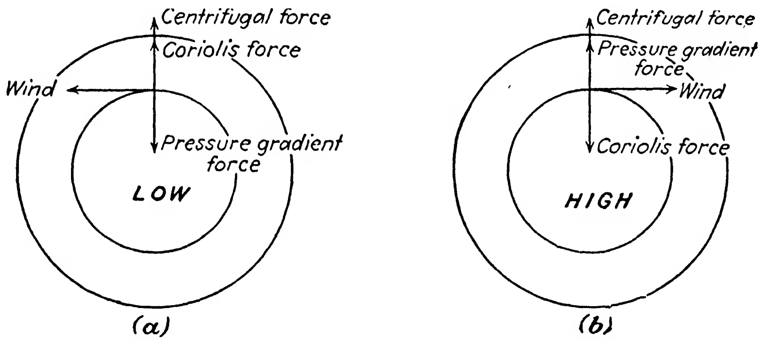

Fig. 43.—Relation between pressure and gradient wind in the Northern Hemisphere.

blows counterclockwise so that the centrifugal force and the Coriolis force counteract the gradient force. A balance of forces would theoretically also be possible with the wind blowing in the other direction. Then the centrifugal force would have to balance the Coriolis force and the gradient force. But with increasing r the centrifugal force would become very small; when $r = \infty$, the motion would not become geostrophic. This alternative is therefore to be rejected.

Similarly, the wind must blow clockwise around a high-pressure area (Fig. 43*b*) so that centrifugal force and gradient force balance the Coriolis force. In order to have the signs in (56.5) right for the anticyclone, it must be noted that $v_\theta < 0$, for the motion is clockwise, and that $\partial p/\partial r < 0$, for the pressure increases outward from the center.

The low-pressure area is also called a *cyclone*, the high-pressure area an *anticyclone*, and the direction of the motion around a

cyclone *cyclonic* and around an anticyclone *anticyclonic.* It will be noted that in the Northern Hemisphere these terms coincide with the terms "counterclockwise" (positive) and "clockwise" (negative).

From (56.5),

$$v = -\omega r \sin \varphi \left(1 \pm \sqrt{1 + \frac{1}{r\omega^2 \sin^2 \varphi} \frac{1}{\rho} \frac{\partial p}{\partial r}}\right) \tag{56.6}$$

The suffix θ has now been omitted. When $\partial p/\partial r = 0$,

$$v = -\omega r \sin \varphi (1 \pm 1)$$

Thus, when the positive sign before the root is taken, an anticyclonic motion is possible even in the absence of a pressure gradient. In this motion the Coriolis force and the centrifugal force balance each other. If the variation of the latitude is disregarded, the motion will be in a circle whose radius

$$r = \left|\frac{v}{2\omega \sin \varphi}\right| \tag{56.7}$$

This circle is called the *circle of inertia.*

Such an inertia motion is dynamically possible. But as r increases, the motion along curved isobars should resemble more and more the geostrophic case. In the geostrophic case the wind velocity vanishes when the pressure gradient is zero. Therefore, in the present case, only the solution with the negative sign before the root in (56.6) has any significance. This choice of only the negative sign in (56.6) is in agreement with the choice of the direction of the wind velocity in Fig. 43; the other direction of v would be represented by the positive root.

In the case of a low-pressure area, $\partial p/\partial r > 0$ and

$$v = +\omega r \sin \varphi \left(\sqrt{1 + \frac{1}{r\omega^2 \sin^2 \varphi} \frac{1}{\rho} \frac{\partial p}{\partial r}} - 1\right) \tag{56.81}$$

The velocity around a low-pressure area is positive, *i.e.*, counterclockwise, in the Northern Hemisphere. In the Southern Hemisphere, it is clockwise, for $\varphi < 0$. Introducing the geostrophic wind velocity,

$$v_g = \frac{1}{2\omega \sin \varphi} \frac{1}{\rho} \frac{\partial p}{\partial r} \tag{53.21}$$

it follows from (56.5) that

$$v_g = v + \frac{v^2}{2\omega \sin \varphi\, r} \tag{56.82}$$

Thus the gradient wind in cyclones is smaller than the geostrophic wind at the same pressure gradient, or for the same wind velocity the pressure gradient is stronger in cyclones than in the case of straight isobars. The reason for this is that in the geostrophic case the gradient force is balanced by the Coriolis force only, whereas in the case of cyclonic gradient wind it is balanced by Coriolis and centrifugal force so that the wind velocity can be smaller, other things being equal.

For high-pressure areas, $\partial p/\partial r < 0$. Upon introducing the absolute value $|\partial p/\partial r|$ of the pressure gradient, it follows that

$$v = -\omega r \sin \varphi \left(1 - \sqrt{1 - \frac{1}{r\omega^2 \sin^2 \varphi} \frac{1}{\rho} \left|\frac{\partial p}{\partial r}\right|}\right) \tag{56.91}$$

The velocity around a high is negative (clockwise) in the Northern Hemisphere. Because the root must be real, it follows that

$$\left|\frac{\partial p}{\partial r}\right| < r\rho\omega^2 \sin^2 \varphi$$

and

$$|v| < \omega r \sin \varphi$$

These conditions are necessary when the balance between pressure-gradient force and centrifugal force on one side and Coriolis force on the other side is to be maintained. If the pressure gradient and therefore the wind velocity become too high, the Coriolis force cannot balance the pressure-gradient force and the centrifugal force, for the latter increases as the square of the velocity. This holds especially near the center of the anticyclone where r is small. Consequently the winds and pressure gradients observed here are generally quite feeble.

According to (56.82) the gradient-wind velocity is absolutely larger in anticyclones than in the geostrophic case (note that v and v_g are now negative) for the same pressure gradient.

Near the equator or in small vortexes, such as tornadoes, where r is small, the Coriolis force is much smaller than the cen-

trifugal force and can be neglected. Then, from (56.5),

$$\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{56.92}$$

This type of motion is called "cyclostrophic." The motion may be clockwise or counterclockwise, but the pressure must be lowest at the center.

When the isobars are not circular, Eq. (56.5) cannot be strictly correct any longer. Nevertheless, it will at least represent an approximation to the actual wind velocity at a given pressure distribution, and this approximation will be better than the geostrophic relation which neglects the effect of curvature. Because v^2/r represents the centrifugal force due to the air motion, one has to substitute for r the radius of curvature of the path of the air, not of the isobar.

The geostrophic wind relation in the case of fairly straight isobars and the gradient-wind relation in the case of appreciably curved isobars give fairly good approximations to the actual wind velocity,[1] especially at levels more than 500 to 1000 m above the ground where the influence of surface friction (see Sec. 76) becomes very slight.

57. Accelerated Motion and a Changing Pressure Field. It cannot be emphasized too strongly that geostrophic wind and gradient wind represent only approximations to the actual relations between pressure gradient and wind. The stronger the accelerations of motion, *i.e.*, the more pronounced the development of the fields of pressure and motion, the greater must be the deviation from the geostrophic or gradient wind.

The effect of the acceleration on horizontal motion will now be discussed in more general terms than in the preceding section where only the centrifugal force was taken into account.

When the motion of the air is accelerated, the pressure-gradient force must balance not only the Coriolis force as in the case of the geostrophic motion but also the acceleration. Therefore, from (47.2),

$$\begin{aligned} \frac{du}{dt} &= 2\omega \sin \varphi\, v - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{dv}{dt} &= -2\omega \sin \varphi\, u - \frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned} \tag{57.1}$$

[1] GOLD, E., *Met. Off.*, No. 109, 1908.

Introducing the geostrophic wind velocity with the components u_g and v_g instead of the pressure gradient,[1]

$$\begin{aligned} \frac{du}{dt} &= 2\omega \sin \varphi \, (v - v_g) = 2\omega \sin \varphi \, v' \\ \frac{dv}{dt} &= -2\omega \sin \varphi \, (u - u_g) = -2\omega \sin \varphi \, u' \end{aligned} \tag{57.2}$$

where u' and v' denote the deviations from the geostrophic wind. It follows that

$$\sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2} = 2\omega \sin \varphi \sqrt{u'^2 + v'^2} \tag{57.3}$$

Thus, the geostrophic deviation is proportional to the acceleration of motion. If α denotes the angle between the x-axis and

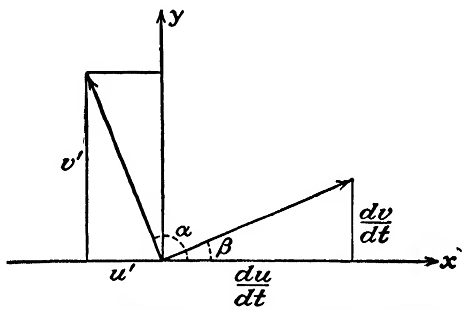

FIG. 44.—Acceleration and geostrophic deviation.

the geostrophic deviation and β the angle between the x-axis and the acceleration (Fig. 44),

$$\tan \beta = \frac{dv/dt}{du/dt} = -\frac{u'}{v'} = -\frac{1}{\tan \alpha}$$

Thus, the acceleration of motion is perpendicular to the geostrophic deviation and to the right of it, as is seen immediately from the signs of the components of the geostrophic deviation and of the acceleration. The study of the deviations from the geostrophic wind field and of the closely related acceleration of motion is of great importance; for, in a purely geostrophic wind field, no change of the surface pressure can occur, as will be shown in the next section.

[1] SUTCLIFFE, R. C., *Quart. J. Roy. Met. Soc.*, **64**, 495, 1938.

We may consider in particular the local derivative of the wind velocity in the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u_g}{\partial t} + \frac{\partial u'}{\partial t}\\ \frac{\partial v}{\partial t} &= \frac{\partial v_g}{\partial t} + \frac{\partial v'}{\partial t}\end{aligned} \tag{57.4}$$

The other terms in the expressions for the acceleration [see (47.1)] will be disregarded. Because u' and v' are, in general, small (for the actual wind approximates the geostrophic wind), their local derivatives will be neglected. But it must be pointed out that this simplification is not necessarily justified, for the comparative smallness of a variable does not always imply comparative smallness of its derivatives. Substituting from the expressions for the geostrophic wind (53.11) and (53.12) and changing the order of differentiation,

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{2\omega \sin \varphi\, \rho} \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial t}\right)\\ \frac{\partial v}{\partial t} &= \frac{1}{2\omega \sin \varphi\, \rho} \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right)\end{aligned} \tag{57.5}$$

$\partial p/\partial t$ is the pressure tendency which can be represented on the weather map by the isallobars, *i.e.*, lines of equal-pressure change. The foregoing equations show that an isallobaric gradient produces an acceleration. According to (57.2), this gives rise to a deviation from the geostrophic wind.

$$\begin{aligned}u' &= -\frac{1}{(2\omega \sin \varphi)^2 \rho} \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right)\\ v' &= -\frac{1}{(2\omega \sin \varphi)^2 \rho} \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial t}\right)\end{aligned} \tag{57.6}$$

u' and v' are here the components of the *isallobaric wind*.[1] The isallobaric wind is proportional to the isallobaric gradient. It blows perpendicular to the isallobars toward the lowest pressure tendency. While the divergence of the geostrophic wind is zero, the isallobaric wind field may have a finite divergence provided that the isallobaric gradient is not the same everywhere. Thus, with regions of rising and falling pressure, convergence

[1] BRUNT, D., and DOUGLAS, C. K. M., *Mem. Roy. Met. Soc.*, **3**, 22, 1928.

and divergence may be associated, and in particular precipitation may be produced by isallobaric convergence.

In the derivation of the isallobaric wind terms the local derivative of the geostrophic deviation and the terms representing the change of the wind field in space have been omitted. This does not imply that these terms are so small that they can, in general, be neglected in comparison with the isallobaric term. But Brunt and Douglas have shown that a consideration of the isallobaric wind component may lead to a better understanding of the weather development where otherwise an explanation may be difficult. The magnitude of the isallobaric term may often amount to 5 m/sec.

According to a statistical investigation of pilot-balloon observations by Möller and Sieber,[1] the isallobaric wind component appears to be directed not perpendicular to the isallobars and toward lower tendency but parallel to the isallobars and so that the region of falling pressure is to the right if one is looking in the direction of the isallobaric wind. Ertel[2] has given a theoretical explanation for this result, based on the assumption that the wind does not follow immediately but only after a certain relaxation time, the changes occurring in the pressure field. But a decision about the average direction of the isallobaric component from such a statistical investigation is subject to considerable error, for the isallobaric component in the statistical material gives very small mean values. Moreover, Möller and Sieber neglect the effect of the curvature of the streamlines. Because clear skies are much more frequent in anticyclones than in cyclones, a preponderance of anticyclonal situations can be expected in the pilot-balloon observations. Berson[3] has shown that the deviation of Möller and Sieber's results from the deductions of Brunt and Douglas can be explained by the assumption that the majority of pilot-balloon ascents were made in anticyclones.

Another investigation into the deviations of the actual wind from the geostrophic wind has been made by Philipps.[4] He

[1] Möller, F., and Sieber, P., *Ann. Hydr.*, **65**, 312, 1937.

[2] Ertel, H., "Methoden und Probleme der dynamischen Meteorologie, Erg. Mathematik und Grenzgebiete," Vol. 5, No. 3, p. 120, Verlag Julius Springer, Berlin, 1938.

[3] Berson, F. A., *Met. Z.*, **56**, 329, 1939.

[4] Philipps, H., *Met. Z.*, **56**, 460, 1939.

found that Brunt and Douglas's theory holds approximately when the isobars are straight lines and equidistant. When the isobars are curved, the expressions (57.6) for the isallobaric wind are less satisfactory, for the other terms come into play which depend on the spatial variation of the wind and pressure field.

58. Divergence, Convergence, and Pressure Variation. The pressure p at any level h represents with a great degree of accuracy the weight per unit area of the total mass of air above this level,

$$p = \int_h^\infty g\rho \, dz$$

Thus the local pressure variation can be expressed by the mass variation

$$\frac{\partial p}{\partial t} = \int_h^\infty g \frac{\partial \rho}{\partial t} \, dz$$

Upon substituting the value for $\partial\rho/\partial t$ from the equation of continuity (47.3), it follows that

$$\frac{\partial p}{\partial t} = -\int_h^\infty g \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right] dz - \int_h^\infty g \frac{\partial(\rho w)}{\partial z} \, dz$$

In the last term on the right-hand side the integration may be carried out, the slight variation of g with the height being neglected. Because the density vanishes at infinity,

$$\frac{\partial p}{\partial t} = -\int_h^\infty g \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right] dz + g(\rho w)_h \tag{58.1}$$

The pressure variation at a given level thus depends on the horizontal-mass transport above the level of observation h and on the vertical-mass transport through the level. It follows in particular for the surface pressure p_0, because the vertical velocity vanishes at the earth's surface, that

$$\frac{\partial p_0}{\partial t} = -\int_0^\infty g \frac{\partial(\rho u)}{\partial x} \, dz - \int_0^\infty g \frac{\partial(\rho v)}{\partial y} \, dz \tag{58.2}$$

If the values for the geostrophic wind are introduced in this equation from (53.11) and (53.12),

$$\frac{\partial p_0}{\partial t} = \int_0^\infty g \frac{\partial}{\partial x}\left(\frac{1}{2\omega \sin\varphi} \frac{\partial p}{\partial y}\right) dz - \int_0^\infty g \frac{\partial}{\partial y}\left(\frac{1}{2\omega \sin\varphi} \frac{\partial p}{\partial x}\right) dz$$

The variation of the Coriolis parameter with the latitude can, in general, be neglected (its effect will be discussed later in Sec. 106); thus, both terms on the right side cancel each other. The pressure variation in a purely geostrophic wind field is zero.

Margules[1] has shown with the aid of Eq. (58.1) that it is impossible to forecast the pressure changes directly from the wind observations, for the wind and its variation in space cannot be determined with sufficient accuracy (see Prob. 18).

Though a direct application of the equation of continuity to a computation of the pressure tendency is impossible, a number of important qualitative results can be derived from it, as has been shown by J. Bjerknes.[2] Equation (58.1) may be developed into the form

$$\frac{\partial p}{\partial t} = -\int_h^\infty g\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right)dz - \int_h^\infty g\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz + g(\rho w)_h \quad (58.3)$$

The first term on the right-hand side now represents the effect of the mass advection; the second term represents the effect of horizontal divergence or convergence of velocity. Bjerknes considers a sinusoidal pressure field whose isobars run in a west-easterly direction, with the lower pressure toward north (Fig. 45*a*). Such a type of pressure distribution is typical for the upper levels of the troposphere in temperate latitudes. The wind velocity, according to the observations, is greater than the speed with which the pressure systems travel. Thus the curvature of the path of the air motion in the neighborhood of AB is cyclonic and around CD anticyclonic. It may be assumed that the wind velocity can be expressed by the gradient-wind relation (56.5). The wind velocity across AB is accordingly given by the equation

[1] MARGULES, M., in F. M. Exner, "Dynamische Meteorologie," 2d ed., p. 75, Verlag Julius Springer, Vienna, 1925.

[2] BJERKNES, J., *Met. Z.*, **59**, 462, 1937. A number of these results have been presented by Bjerknes in a series of lectures given at the Meteorological Office, Toronto, in August, 1939. A revised issue of these lectures is available there.

$$v_{AB} = \frac{1}{\rho}\frac{1}{2\omega \sin \varphi}\frac{\partial p}{\partial r} - \frac{1}{2\omega \sin \varphi}\frac{v_{AB}^2}{r} \tag{58.4}$$

and across CD by

$$v_{CD} = \frac{1}{\rho}\frac{1}{2\omega \sin \varphi}\frac{\partial p}{\partial r} + \frac{1}{2\omega \sin \varphi}\frac{v_{CD}^2}{r} \tag{58.5}$$

The wind velocity across the trough line AB is less than the wind velocity across the ridge CD. If it is assumed further that the wind blows parallel to the isobars, there is divergence in the area $ABCD$ bounded by the straight lines AB and CD and by the isobars AD and BC. Consequently the pressure must fall in

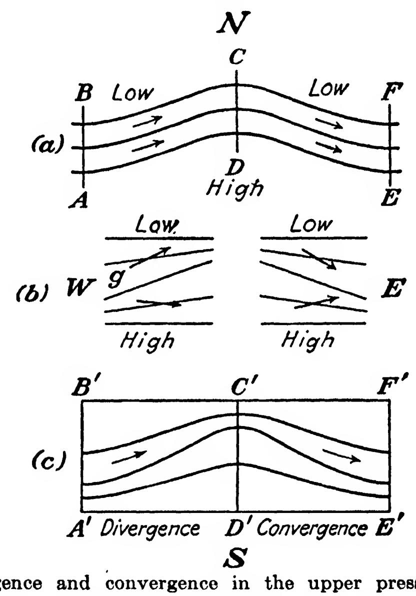

FIG. 45.—Divergence and convergence in the upper pressure field. (*After J. Bjerknes.*)

this area and the trough AB wanders toward east. By similar reasoning, it can be seen that the pressure rises in the area $CDEF$, owing to convergence, so that the ridge also moves eastward.

In reality the isobaric field generally has not the regular appearance shown in Fig. 45*a*. It resembles rather the pressure distribution shown in Fig. 45*c*, with a central isobar of maximum amplitude. The amplitudes decrease northward and southward, and straight isobars limit the pressure system.

There are now regions between the parallel straight isobars where the isobars converge and diverge, as is shown separately in Fig. 45*b*. A wind blowing in the direction of converging isobars is subjected to an increasing pressure gradient. Consider a mass of air originally at G. If the mass is here under balanced forces, the Coriolis force, which is directly proportional to the wind velocity, balances the pressure gradient. As the mass proceeds, it comes into regions of stronger pressure gradient though retaining its previous velocity owing to inertia. Thus, the pressure-gradient force will be larger than the Coriolis force, and the wind is deflected across the isobars toward lower pressure. Similarly, it is easily seen that a wind moving in the direction of diverging isobars is deflected toward the higher pressure. The effect of this deviation of the wind from the geostrophic balance makes itself felt, also, in an isobaric configuration of the type represented in Fig. 45*c*. Bjerknes assumes that there is no transport across the straight isobars $B'C'F'$ and $A'D'E'$. In the region $A'D'C'B'$, divergence of horizontal motion occurs owing to the velocity across the ridge $C'D'$ being faster than across the trough $A'B'$. The pressure falls in front of the trough, and the trough moves eastward. Similarly, convergence occurs in the region $C'D'E'F'$ so that the pressure rises here and the ridge also moves eastward. The motion of the pressure systems caused in this manner may be called *the cyclostrophic effect*, for it is due to the difference of the cyclostrophic terms in cyclonic and anticyclonic motion. These considerations explain why ridges and troughs wander eastward in a westerly current. The motion is not due to a bodily transport of the pressure systems in the current. Similar reasoning can also be applied to the closed isobaric systems that are observed in the lower atmospheric layers.

The preceding analysis is, of course, based on the assumption that the wind follows the gradient-wind relation. If this were not the case, the position of the regions of convergence and divergence might be quite different; but the agreement with the observed eastward motion shows that the assumption represents at least a good approximation to the facts. A more detailed study of the horizontal divergence of motion in relation to the motion of pressure centers has recently been published by Petterssen.[1]

[1] Petterssen, S., *Quart. J. Roy. Met. Soc.*, **66**, suppl., 102, 1940.

The parts of the isobars in Fig. 45*a* lying near *AB* are farther south than the parts near *CD*. Therefore, the first term on the right-hand side of (58.4) will be larger than the corresponding term in (58.5). If the second, centrifugal terms were to be disregarded, the wind across *AB* and *EF* would therefore be greater than the wind across *CD*. This difference would give rise to a motion of the pressure systems westward, which may be called the *latitudinal effect*.[1] For pressure systems with a large north-southerly extent the latitudinal effect becomes large and may be greater than the cyclostrophic effect. Such disturbances will be studied in Sec. 106. In the ordinary disturbances of the temperate latitudes that are associated with the migrating cyclones, however, the cyclostrophic effect is preponderant, for its magnitude does not depend on the north-southerly amplitude but on the radius of the curvature of the trajectory only.

59. Pressure Distribution in a Moving Cyclone. Exner[2] has discussed the effect of the superposition of a uniform straight motion on a circular motion. Starting with the field of motion the pressure field can be derived from the equations of motion. Exner's calculation is an interesting example of the determination of the pressure field from the wind field, a procedure that appears in many respects more satisfactory than the determination of the wind distribution from the pressure distribution under the assumption of a balance of forces.

The x-axis may be chosen parallel to the straight motion U (Fig. 46). The fluid may be regarded as incompressible which is not a serious restriction in the case of horizontal motion.

The center of the cyclonic motion O moves with the velocity U along the x-axis. Let r be the distance of a point P from O and ζ the angular velocity of cyclonic motion which may at first be an arbitrary function of r. Then

$$r^2 = (x - Ut)^2 + y^2$$

Further

$$\begin{aligned} u &= -r\zeta \sin \theta + U = -\zeta y + U \\ v &= r\zeta \cos \theta = \zeta(x - Ut) \end{aligned} \tag{59.1}$$

It may be noted that the equation of continuity is satisfied.

[1] Rossby, C.-G., and collaborators, *J. Mar. Research*, **2**, 38, 1939.

[2] Exner, F. M., "Dynamische Meteorologie," 2d ed., p. 268, Verlag Julius Springer, Vienna, 1925.

From (47.2), it follows that

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\omega \sin\varphi\, v &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\omega \sin\varphi\, u &= -\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned} \tag{59.2}$$

From these equations the pressure gradient can be determined.

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\zeta^2(x - Ut)\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\zeta^2 y\end{aligned}$$

which is the centrifugal acceleration. It follows that

$$\begin{aligned}-\frac{1}{\rho}\frac{\partial p}{\partial x} &= -\zeta(x - Ut)(\zeta + 2\omega \sin\varphi)\\ -\frac{1}{\rho}\frac{\partial p}{\partial y} &= -\zeta y(\zeta + 2\omega \sin\varphi) + 2\omega \sin\varphi\, U\end{aligned} \tag{59.3}$$

The pressure gradient is a superposition of the gradient corresponding to the circular motion on the geostrophic gradient and could have been written down directly.

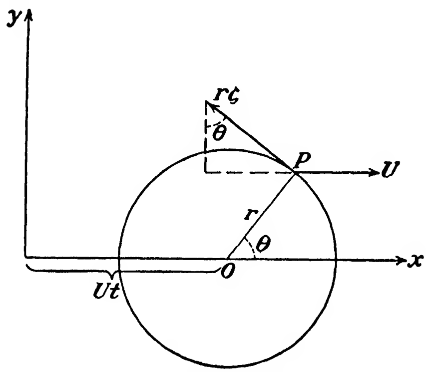

Fig. 46.—Superposition of straight and circular motion.

Let V be the absolute value of the wind velocity and α the angle that it makes with the x-axis. $\partial p/\partial n$ is the absolute value of the pressure gradient and β the angle that it makes with the x-axis. Then

$$-\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) = -\frac{1}{\rho}V\frac{\partial p}{\partial n}\cos(\alpha - \beta) = -U\zeta^2(x - Ut) \tag{59.4}$$

This relation shows that the angle between pressure gradient and wind is now not 90°. The angle would be 90° if $U = 0$ or if $\zeta = 0$, *i.e.*, if a pure gradient motion or a pure geostrophic motion exists.

To show the deviation of this comparatively simple motion from the balanced motion more in detail, Exner assumes that the cyclonic velocity $r\zeta$ is inversely proportional to the distance from O,

$$r\zeta = \frac{a}{r} \tag{59.5}$$

a velocity distribution that has been found to hold in the outer part of some tropical and many extratropical[1] cyclones. Equation (59.5) implies that the angular momentum relative to the

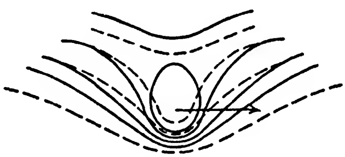

FIG. 47.—Isobars (full lines) and streamlines (broken lines) in a moving vortex. (*After F. M. Exner.*)

earth remains constant. Very close to the center, this velocity distribution cannot hold, for it would lead to an infinitely high velocity at the center. The streamlines are given by

$$u\,dy - v\,dx = 0$$

Substitution of (59.1) and integration show that the streamlines satisfy the equation

$$a \ln r - Uy = \text{const} \tag{59.6}$$

From (59.3) the equation of the isobars follows,

$$\frac{p}{\rho} = a2\omega \sin \varphi \ln r - \frac{a^2}{2r^2} - 2\omega \sin \varphi\, Uy + \text{const}$$

Figure 47 shows streamlines (broken) and isobars (full) computed by Exner with the aid of these equations. It is assumed that $U/a = 0.4$ so that the velocity of translation is four-tenths the circular velocity at the distance 1 from O.

[1] GOLDIE, A. H. R., *Met. Off., Geophys. Mem.*, No. 72, 1937.

Problems

13. A geostrophic wind scale gives the geostrophic wind velocity corresponding to any pressure gradient as shown on the weather map, as a function of the distance between two consecutive isobars. How can such a geostrophic wind scale be used to find the thermal wind when the isotherms are plotted? To simplify matters, it may be assumed that the temperature does not change with the altitude.

14. Show that the geostrophic wind is independent of the height in a barotropic atmosphere.

15. Express the geostrophic wind velocity by the geopotential and the temperature distribution in a surface of equal potential temperature (isentropic surface).

16. Find the equation of the isobaric surfaces in a cyclonic circular vortex whose inner part ($r < R$) rotates with a constant angular velocity η, the velocity of the outer part ($r > R$) being inversely proportional to the distance from the center (Rankine vortex) when the velocity is continuous across the circle of the radius R.

17. How may a geostrophic wind scale be used to find the isallobaric wind?

18. How accurately must the horizontal change of the wind velocity be known in order to predict a surface-pressure change of 1 mb/3 hr with the aid of the equation of continuity? For the sake of simplicity, it may be assumed that the horizontal wind velocity and direction do not change with the elevation and that the wind direction and the density are constant in horizontal direction.

19. How accurately must the horizontal change of the wind velocity be known in order to determine vertical motions of at least 10 cm/sec from the horizontal wind velocity? Make the same simplifying assumptions as in Prob. 18, and assume a steady field ($\partial\rho/\partial t = 0$).

CHAPTER VIII

SURFACES OF DISCONTINUITY

60. General Expression for Surfaces of Discontinuity. Frequently, air masses from different regions and therefore of different temperatures are brought close together so that a zone of rapid temperature transition is formed (see also Sec. 108). Owing to the small scale of the maps and cross sect ons that must be used in meteorological practice, these zones of transition appear mostly as sharp surfaces of discontinuity and may be treated as such mathematically. At the surface of discontinuity the rapid but gradual variation of the temperature in a finite interval may then be replaced by a sudden change through the surface of discontinuity.

When the variable considered, *e.g.*, the temperature, shows a discontinuity, the surface is called a *discontinuity of zero order* with respect to this variable (see Fig. 52*a*, page 176). When the variable is continuous but its first derivative is discontinuous across the surface of discontinuity, we have a *discontinuity of the first order*, and so on for higher derivatives (see Fig. 52*b*, page 176). We shall see below that the pressure is continuous but its first derivatives are discontinuous. Thus, the surfaces of discontinuity are discontinuities of the first order with respect to the pressure but of zero order with respect to the temperature or to the density.

Because the air masses differ in temperature, their densities must in general be different too. The surface of discontinuity must therefore have a certain inclination with respect to the surface of the earth which will now be determined. The equation for the surface of discontinuity may be defined by the condition that the pressure must here remain continuous.[1] If the pressure p_1 in the first air mass measured at a point P on the surface of discontinuity were different from the pressure p_2* in the second

[1] Bjerknes, V., *Geofys. Pub.*, **2**, No. 4, 1921.

* In the following, the index 1 will be used to refer to the colder mass and the index 2 to refer to the warmer mass.

air mass measured at the same point on the surface of discontinuity, the pressure gradient would be infinite. This is obviously impossible. Thus

$$p_1 - p_2 = 0 \tag{60.1}$$

Upon differentiating, it follows that

$$d(p_1 - p_2) = \left[\left(\frac{\partial p}{\partial x}\right)_1 - \left(\frac{\partial p}{\partial x}\right)_2\right] dx + \left[\left(\frac{\partial p}{\partial y}\right)_1 - \left(\frac{\partial p}{\partial y}\right)_2\right] dy + \left[\left(\frac{\partial p}{\partial z}\right)_1 - \left(\frac{\partial p}{\partial z}\right)_2\right] dz = 0 \tag{60.2}$$

Equation (60.2) states that the component of the pressure gradient parallel to the surface of discontinuity must be the same in the first and second air mass when measured directly at the surface of discontinuity. This condition gives the differential equation for the surface of discontinuity. According to (60.2) the inclination of the surface of discontinuity to the horizontal xy-plane in the x- and y-direction is given, respectively, by

$$\frac{dz}{dx} = -\frac{(\partial p/\partial x)_1 - (\partial p/\partial x)_2}{(\partial p/\partial z)_1 - (\partial p/\partial z)_2} \tag{60.31}$$

$$\frac{dz}{dy} = -\frac{(\partial p/\partial y)_1 - (\partial p/\partial y)_2}{(\partial p/\partial z)_1 - (\partial p/\partial z)_2} \tag{60.32}$$

The intersection between the surface of discontinuity and the ground that may be identified with the xy-plane is the *front*. The front makes an angle with the x-axis whose tangent is given by

$$\frac{dy}{dx} = -\frac{(\partial p/\partial x)_1 - (\partial p/\partial x)_2}{(\partial p/\partial y)_1 - (\partial p/\partial y)_2} \tag{60.33}$$

Frequently, it will be convenient to orient the coordinate system so that the front runs parallel to the y-axis. Then $dy/dx = \infty$ and

$$\left(\frac{\partial p}{\partial y}\right)_1 - \left(\frac{\partial p}{\partial y}\right)_2 = 0 \tag{60.4}$$

According to this equation the horizontal component of the pressure gradient parallel to the surface of discontinuity is the same on both sides, which is a special case of the condition stated by (60.2). The inclination in the y-direction vanishes, of course, with this orientation of the coordinate system.

61. The Pressure Distribution at Fronts. It may be assumed, as at the end of the preceding section, that the front is parallel to the y-axis so that

$$\left(\frac{\partial p}{\partial y}\right)_1 - \left(\frac{\partial p}{\partial y}\right)_2 = 0 \tag{60.4}$$

The index 1 denotes again the colder mass, and the index 2 the warmer mass.

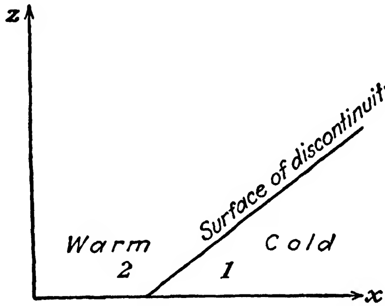

FIG. 48.—Position of cold and warm air at the surface of discontinuity.

The colder air may extend in the direction of increasing x, and the warmer air toward decreasing x (Fig. 48). Because the cold air must be under the warmer air, dz/dx must be positive. Now

$$\left(\frac{\partial p}{\partial z}\right)_1 < \left(\frac{\partial p}{\partial z}\right)_2 \tag{61.1}$$

for the pressure decreases faster with the height in the cold mass than in the warm mass. Hence, the condition that $dz/dx > 0$ implies, according to (60.31), that

$$\left(\frac{\partial p}{\partial x}\right)_1 > \left(\frac{\partial p}{\partial x}\right)_2 \tag{61.2}$$

Equations (60.4) and (61.2) state that the component of the pressure gradient parallel to the front is the same on both sides of the front, whereas the component normal to the front is greater in the cold mass than in the warm mass.

It follows that the isobars at the front must be bent so that they form a trough of low pressure. This can most easily be seen by considering Fig. 49. If $(\partial p/\partial x)_1$ and $(\partial p/\partial x)_2$ are both

positive (Fig. 49*a*), the total horizontal gradient $(\partial p/\partial n)_1$ makes a smaller angle with the x-axi —*i.e.*, with the direction normal to the front—than $(\partial p/\partial n)_2$ does. The direction of the isobars (broken lines) changes at the front so that a trough of low pressure must exist. If $(\partial p/\partial x)_2$ is negative and $(\partial p/\partial x)_1$ is positive, the pressure trough is even more pronounced (Fig. 49*b*). The reader will have no difficulty in proving that the rule holds when both $(\partial p/\partial x)_1$ and $(\partial p/\partial x)_2$ are negative. In this case, $(\partial p/\partial x)_1$ must be absolutely smaller than $(\partial p/\partial x)_2$ according to (61.2).

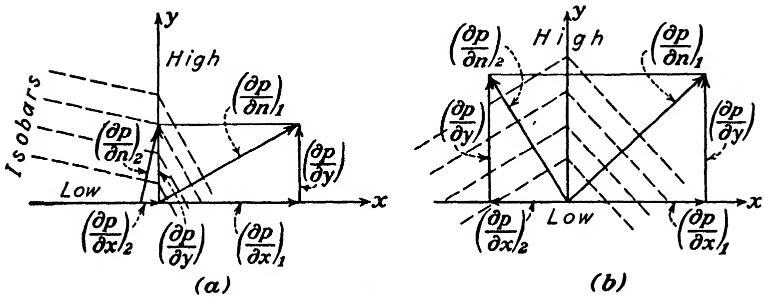

FIG. 49.—The pressure trough at fronts.

The proof for the existence of a pressure trough is obviously independent of the orientation of the coordinate system and could have been given in the same manner if the position of the cold and warm masses were interchanged, for in this case $dz/dx < 0$.

62. Surfaces of Discontinuity in a Geostrophic Wind Field. When the field of motion is geostrophic, the inclination and orientation of the surface of discontinuity can be expressed by the temperature and the wind at the surface of discontinuity.

As before, it is convenient to choose the coordinate system so that the y-axis is parallel to the front. The x-axis makes, then, an angle β with the direction toward the east. The equations for geostrophic wind are, according to (53.11), (53.12), and (45.8),

$$-2\omega \sin \varphi \, v = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$2\omega \sin \varphi \, u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-2\omega \cos \varphi(u \cos \beta - v \sin \beta) + g = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Upon substituting from the first of these equations in (60.4), it follows that

$$\rho_1 u_1 = \rho_2 u_2 \tag{62.1}$$

According to (61.2),

$$\rho_1 v_1 > \rho_2 v_2 \tag{62.11}$$

From the condition (62.11), it is seen that, in general, $v_1 > v_2$, for the ratio ρ_1/ρ_2 is generally not very different from unity. Thus, if one looks in the direction from the cold to the warm air (Fig. 50), the warm air must move more slowly toward the right or faster to the left than the cold air. The rule may be remembered more easily in the form that the air mass toward which the observer is looking must move faster to the left than the other air mass. But it should be kept in mind that this rule has been derived for unaccelerated motion only and that it may break down when sufficiently strong deviations from the geostrophic wind occur. Because, in general, these deviations are small, however, the rule will often be found valid in the atmosphere.

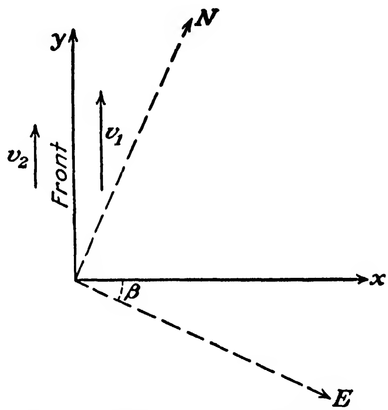

FIG. 50.—Relative wind velocities at a front in a geostrophic wind field.

The inclination of the surface of discontinuity to the horizontal

$$\begin{aligned}\tan \alpha_0 &= \frac{dz}{dx} \\ &= + \frac{2\omega \sin \varphi(v_1\rho_1 - v_2\rho_2)}{2\omega \cos \varphi \sin \beta(v_1\rho_1 - v_2\rho_2) + g(\rho_1 - \rho_2)}\end{aligned} \tag{62.2}$$

according to (60.31), where α_0 is the angle between the direction tangential to the surface of discontinuity and the horizontal in the geostrophic case. The first term in the denominator on the right-hand side of (62.2) is, in general, much smaller than the second term, as follows from the considerations on page 145. It may therefore be omitted except when $\rho_1 = \rho_2$. In this case

only the wind component parallel to the surface of discontinuity is discontinuous and

$$\tan \alpha_0 = \frac{\tan \varphi}{\sin \beta} \tag{62.3}$$

The direction cosines of the earth's axis are $\cos \varphi \sin \beta$, $\cos \varphi \cos \beta$, and $\sin \varphi$ (page 125); and the direction cosines of the normal to the surface of discontinuity are $-\sin \alpha_0$, 0, and $\cos \alpha_0$. Hence, the cosine of the angle between the normal to the frontal surface and the earth's axis is

$$-\cos \varphi \sin \beta \sin \alpha_0 + 0 \cos \varphi \cos \beta + \sin \varphi \cos \alpha_0$$

It will be seen from (62.3) that this expression vanishes so that the two directions are at right angles. A surface along which the wind only is discontinuous is therefore parallel to the earth's axis provided that the motion is geostrophic. When $\beta = 0$ so that the x-axis coincides with the west-east direction and the front therefore with the north-south direction, $\alpha_0 = \pi/2$. The surface of discontinuity is vertical. When $\beta = \pi/2$, so that the front is in the west-east direction, $\alpha_0 = \varphi$.

When $v_1 = v_2$,

$$\tan \alpha_0 = \frac{2\omega \sin \varphi v}{2\omega \cos \varphi \sin \beta v + g} \tag{62.4}$$

According to (54.1), this is also the inclination of the isobaric surfaces. The inclination of a surface of density discontinuity is therefore equal to the inclination of an isobaric surface.

In the general case when $\rho_1 \neq \rho_2$ and $v_1 \neq v_2$, the first term in the denominator of (62.2) may be neglected in comparison with the second term and the temperature may be introduced instead of the density by means of the gas equation (4.1). When the air is moist, the variation of the density due to the moisture content can be taken into account by the introduction of the virtual temperature [see Eq. (5.3)] instead of the temperature. Because the pressure is the same on both sides of the surface of discontinuity,

$$\tan \alpha_0 = \frac{2\omega \sin \varphi}{g} \frac{T_2 v_1 - T_1 v_2}{T_2 - T_1} \tag{62.5}$$

In this simple form the inclination of a surface of discontinuity was first derived by Margules.[1]

[1] MARGULES, M., *Met. Z.*, *Hann-Vol.*, 293, 1906.

Introducing the temperature and wind discontinuities,

$$\Delta T = T_2 - T_1$$
$$\Delta v = v_2 - v_1$$
$$\tan \alpha_0 = \frac{2\omega \sin \varphi}{g}\left(v_1 - T_1 \frac{\Delta v}{\Delta T}\right) \tag{62.6}$$

where the inclination of the surface of discontinuity is expressed by the inclination of the isobaric surfaces in the cold mass plus an additional term depending on the temperature and wind discontinuities. Because the first term in the parenthesis is, in general, much smaller than the second term, the inclination of the isobaric surfaces is much less than the inclination of the surfaces of discontinuity. For $T_1 = 273°$, $\Delta T = 10°\text{C}$, $v_2 = 0$ m/sec, it is at 42.4° latitude, as the table below indicates,

	when $v_1 = 10$ m/sec	when $v_1 = 20$ m/sec	when $v_1 = 30$ m/sec
Inclination of the isobaric surfaces..........	1/10,000	1/5000	1/3333
Inclination of the surfaces of discontinuity...	1/353	1/176	1/118

63. Accelerations at Frontal Surfaces. It will be assumed again that the y-axis coincides with the front and that the angle between the x-axis and the east direction is β. In order to express the orientation of a surface of discontinuity in an arbitrary wind field by the velocities, accelerations, and temperatures on both sides of the surface, the components of the pressure gradients in (60.31) and (60.4) have to be expressed by the values following from (47.2). The complete equations that would result from this substitution are very complicated and cannot be discussed satisfactorily, for the vertical velocities cannot be determined directly. In practice, however, it is permissible to neglect the Coriolis term containing the vertical velocity in the first Eq. (47.2) for the horizontal pressure gradients, for the vertical velocity components are much smaller than the horizontal components. In the third equation of (47.2) the terms on the left-hand side may be neglected in comparison with the acceleration of gravity. From the discussion on page 171, it follows that this approximation is justified as long as the temperature dis-

continuity does not become too small, *i.e.*, as long as the front is of practical importance and not too weak. According to (60.4) and (60.31) the following two relations must then be satisfied:

$$\rho_1\dot{v}_1 - \rho_2\dot{v}_2 + 2\omega \sin \varphi(u_1\rho_1 - u_2\rho_2) = 0 \tag{63.1}$$

$$\rho_1\dot{u}_1 - \rho_2\dot{u}_2 = 2\omega \sin \varphi(v_1\rho_1 - v_2\rho_1) - g(\rho_1 - \rho_2) \tan \alpha \tag{63.2}$$

These equations hold also when the surface of discontinuity is moving with a constant velocity provided that the velocity

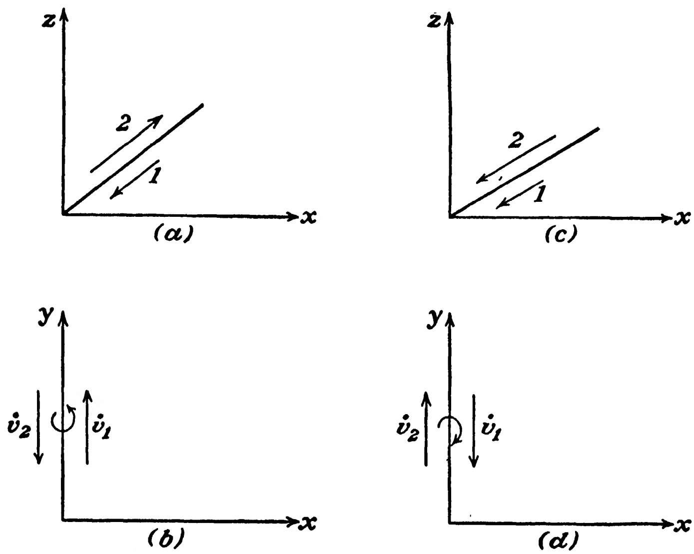

FIG. 51.—Acceleration at surfaces of discontinuity. (*After J. Bjerknes.*)

components are measured in a coordinate system moving with the velocity of the frontal surface. The expressions u_1, v_1, u_2, and v_2 represent, then, the differences between the velocity of the frontal surface and of the cold and warm air masses.

When the warm air ascends at the surface of discontinuity while the cold air descends or ascends more slowly (Fig. 51*a*), the cold air must move more slowly in the horizontal than the warm air, $u_1 < u_2$, or, with sufficient accuracy, $\rho_1 u_1 < \rho_2 u_2$; for the ratio $\rho_1 : \rho_2$ is approximately equal to 1. It follows[1] from (63.1) that

$$\rho_1\dot{v}_1 > \rho_2\dot{v}_2 \tag{63.3}$$

[1] BJERKNES, J., *Geofys. Pub.*, **3**, No. 6, 1924. See also E. PALMÉN, *Soc. Sci. Fenn. Comm. Phys. Math.*, **4**, No. 20, 1928.

The accelerations tangential to the surface tend to produce a cyclonic vorticity (Fig. 51*b*).

If the cold air has an ascending motion while the warm air descends or ascends more slowly or if the cold air descends more slowly than the warm air, (Fig. 51*c*), $\rho_1 u_1 > \rho_2 u_2$ and, according to (63.1),

$$\rho_1 \dot{v}_1 < \rho_2 \dot{v}_2 \tag{63.4}$$

In this case the tangential accelerations tend to produce anticyclonic vorticity.

With the aid of aerological ascents, it is sometimes possible to determine the inclination of a surface of discontinuity, $\tan \alpha$, directly and to compute the inclination $\tan \alpha_0$ that the surface would have in the geostrophic case. When $\tan \alpha_0$ is introduced in (63.2) with the aid of (62.5), the small Coriolis term in the denominator being omitted,

$$\rho_1 \dot{u}_1 - \rho_2 \dot{u}_2 = g(\rho_1 - \rho_2)(\tan \alpha_0 - \tan \alpha) \tag{63.5}$$

When the surface of discontinuity is inclined more steeply than in the geostrophic case, $\alpha > \alpha_0$,

$$\rho_1 \dot{u}_1 < \rho_2 \dot{u}_2$$

The horizontal acceleration of the air perpendicular to the surface of discontinuity is then greater in the warm air than in the cold air (the effect of the different densities of both air masses may be disregarded). If u_1 and u_2 were originally not very different from each other, it follows that the warmer air will eventually move faster and ascend over the colder air if the motion is in the direction from the warm air to the cold air (warm front). If the motion is in the opposite direction as in the case of a cold front, subsidence will take place.

When the surface of discontinuity is less steeply inclined than in the geostrophic case, the vertical motions in the cold and in the warm air are reversed.

Owing to the rapid change of the temperature through the frontal surface the number of solenoids becomes very great here, as has been shown by Bergeron and Swoboda.[1] Therefore, the circulation acceleration near the frontal surface must be very

[1] BERGERON, T., and SWOBODA, G., *Veröffentlich. Geophys. Inst. Leipzig*, 2d Ser., **3**, 63, 1924–1927.

strong [see Eq. (52.2)], which explains the importance of fronts for the dynamics of the weather in general.

64. Zones of Transition. In the preceding formulas a sharp discontinuity is always assumed at the surface of discontinuity. If the mixing is very strong, however, the transition from one air mass to the other may become so gradual that the finite width of the zone of transition has to be taken into account. The isotherms may run across this zone of transition somewhat as shown in Fig. 52*b*. Here the zone of transition is separated from

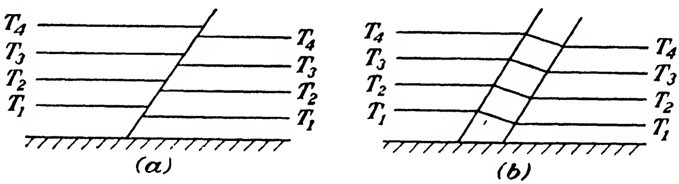

FIG. 52.—(*a*) Isotherms at a sharp surface of discontinuity, (*b*) isotherms at a diffuse surface of discontinuity.

the two air masses by discontinuities of the first order with respect to the temperature, but the discontinuity may very well be of a higher order. A mathematical discussion of the position of these zones of transition will not be given here. It is sufficient to say that in the equations of the foregoing sections the finite differences of quantities (or products of quantities) measured across the front have to be replaced by their horizontal derivatives across the zone.[1]

65. Fronts and Pressure Tendencies. When a frontal surface passes over a station, the pressure may be expected to change, for air of a given temperature is replaced by air of different temperature. This "static" effect of the frontal passage on the recorded surface pressure can easily be computed approximately.[2] Consider, for instance, the

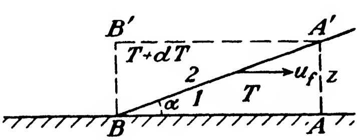

FIG. 53.—Static effect of the frontal passage on the surface pressure.

passage of a cold-front surface (Fig. 53) whose inclination is tan α and whose velocity is u_f. Both quantities may be constant. The mean temperature in the cold mass may be T and in the warm

[1] BJERKNES, *loc. cit.*, PALMÉN, *loc. cit.*

[2] GIÃO, A., *Mém. Off. Nat. Mét. France*, No. 20, 104, 1929.

mass $T + dT$. According to (6.21) the pressure at A

$$p_0 = pe^{\frac{g}{R}\frac{z}{T}} \tag{6.21}$$

where p is the pressure at A' on the surface of discontinuity. When the front has moved the distance AB, the warmer air column is above the point of observation. The surface pressure changes from p_0 to $p_0 + dp_0$, owing to the replacement of the cold air by the warmer air, whereas the pressure at z remains unchanged. From (6.21), it follows by logarithmic differentiation that

$$\frac{dp_0}{p_0} = -\frac{g}{R}\frac{z}{T^2}\,dT \tag{65.1}$$

dp_0 is the pressure variation during the time that it takes the front to travel the distance AB. Because $z = AB \tan \alpha$, this time interval is $z/\tan \alpha\, u_f$. Upon dividing the right-hand side of (65.1) by this value the pressure variation per unit time is found to be

$$\frac{\partial p_0}{\partial t} = -\frac{g}{R}\tan \alpha\, u_f p_0 \frac{dT}{T^2} \tag{65.2}$$

For practical purposes it may be assumed that $p_0 = 1000$ mb and $T = 273°$ and that the tendency $\partial p_0/\partial t$ is expressed in millibars per 3 hr. When the wind velocity is given in m/sec and the temperature difference in degrees centigrade, (65.2) may be written in the form

$$\frac{\partial p_0}{\partial t} = -5 \tan \alpha\, u_f\, dT \tag{65.3}$$

For a cold front the same expression holds except that u_f or dT must be given the opposite sign.

Equation (65.3) can readily be evaluated with the aid of the diagram (Fig. 54) constructed by Mr. T. J. G. Henry, provided that the necessary data are known. Let the velocity of the front be 30 mph, its slope $1/100$, and the temperature 10°F. To find the pressure tendency, ascend vertically from the scale on the left marked 30 mph to the intersection with the slope line $1/100$, then move horizontally to the right to the intersection with the temperature-difference line 10°F. Upon moving vertically down-

ward from here the pressure tendency is found to be not quite 4 mb/3 hr.

In practice the static effect of the frontal passage is, in general, superimposed on pressure changes extending over a large area on both sides of the front. Then $\partial p_0/\partial t$ represents the difference between the pressure tendencies on both sides of the front.

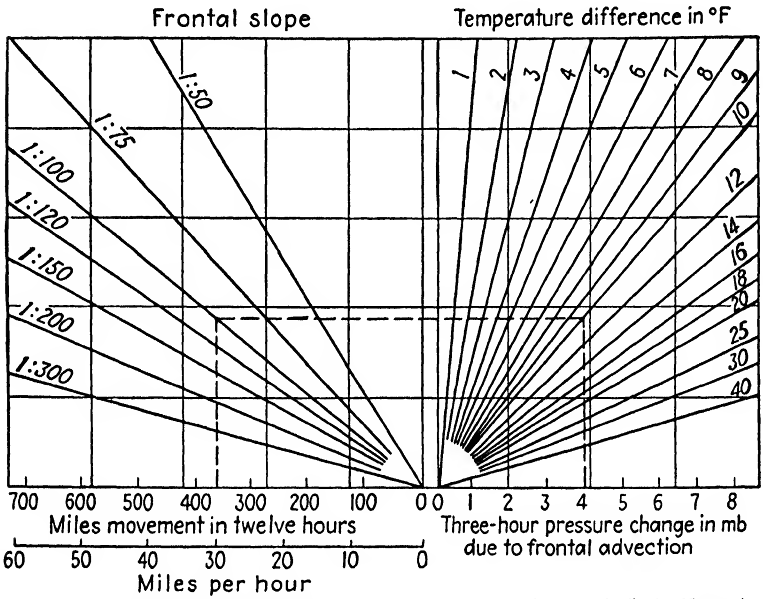

FIG. 54.—Pressure change due to frontal advection. (*After T. J. G. Henry.*)

It should be clearly understood that (65.3) gives only the static pressure effect and that the difference of the pressure tendencies on both sides of the front will in general not agree with the values computed from the above formulas, for other effects, such as vertical motions, convergence and divergence near the frontal surface, and pressure variations at higher levels, are superimposed (page 326). The practical value of the calculation consists rather in the possibility of estimating the relative importance of the static and dynamic influences on the variation of the pressure tendencies. A computation of the static effect will show that frequently the simple concept that the approach of cold air makes the pressure rise or that the approach of warm air makes it

fall is incorrect, or at least incomplete, for it may often not account for the total variation of the pressure tendency.

Problems

20. Find an expression for the orientation of a front (angle between front and direction toward east) in a geostrophic wind field when the east and north components of the wind and the temperature on both sides of the front are known.

21. Find the expression for the inclination of a surface of discontinuity in a gradient-wind field with circular isobars.

CHAPTER IX

KINEMATICAL ANALYSIS OF THE PRESSURE FIELD

66. The Motion of Characteristic Curves. An attempt to compute the future weather by direct application of the equations of thermodynamics and dynamics seems at present not promising, owing to the complexity of the problem. Exner[1] has studied the feasibility of this line of attack, but the simplifications that have had to be introduced are so many that the results obtained do not appear very encouraging. A similar attempt on a much broader basis has been undertaken by Richardson.[2] His work shows very clearly that, in the present state of our knowledge, this way is not practical. The thermodynamics and dynamics of the atmosphere enable us to understand the weather processes and to obtain numerical results in some cases where it is possible to single out a phenomenon influenced by a limited number of factors, such as adiabatic ascent of moist air or motion under balanced forces. But a computation of the future weather by dynamical methods will be possible only when it is known more definitely which factors have to be taken into account under given conditions and which may be neglected.

It is, however, possible to arrive at quantitative formulas of direct use to the forecaster by another method developed largely by Petterssen.[3] This method is based on an analysis of the pressure field and its variations. The pressure has been chosen because it is the element that can be measured most accurately and that is much less affected by local disturbances than, for instance, temperature or wind. Moreover, a determination even of the future pressure distribution only is of great assistance to the forecaster.

[1] Exner, F. M., *Sitz.-Ber. Akad. Wiss. Wien*, II*a*, **115**, 1171, 1906; **116**, 995, 1907; **119**, 697, 1910.

[2] Richardson, L. F., "Weather Prediction by Numerical Process," Cambridge University Press, London, 1922.

[3] Petterssen, S., *Geofys. Pub.*, **10**, No. 2, 1933.

Petterssen's method makes use not of dynamical but of kinematical principles. Nevertheless, it is of interest also in dynamic meteorology—apart from its practical value—for it relates quantities of dynamic importance, as, for instance, pressure tendencies and pressure gradients. In this chapter, some of the basic features of Petterssen's method will be discussed briefly, but for the technique of the synoptic application of the following formulas the reader is referred to the original literature and especially to Petterssen's book "Weather Analysis and Forecasting."

Lines of equal pressure, equal-pressure tendency, or, in general, lines along which a derivative or the sum of derivatives of the pressure is constant may be called *characteristic curves.* It is convenient to introduce the notation

$$p_{l,m,n} = \frac{\partial^{l+m+n}p}{\partial x^l\, \partial y^m\, \partial t^n} \tag{66.1}$$

The subscript l indicates the number of differentiations with respect to x, m the number with respect to y, and n the number with respect to t. The characteristic curves whose motions are to be studied will, as a rule, be of the form

$$p_{l,m,n} = \text{const} \tag{66.2}$$

An isobar, for instance, is represented by the condition that p_{000} = const.

Let S_0 (Fig. 55) be the position of a characteristic curve at a time t and S_1 its position at a time $t + dt$, and choose an arbitrary axis L pointing in the direction of motion of the curve. The velocity of the curve in the direction of L

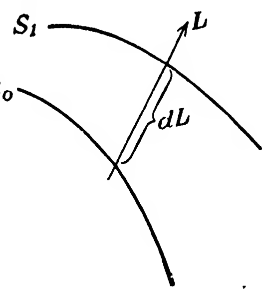

FIG. 55.—Velocity of a characteristic curve.

$$c = \frac{dL}{dt}$$

Frequently, it will be convenient to choose L at right angles to S and as x-axis.

Consider now a fictitious particle forced to remain on a characteristic curve $p_{l,m,n}$ = const, which moves with the velocity c of this line in the direction of L. The direction may be chosen

as the x-axis. The individual variation $dp_{l,m,n}/dt$, for this particle must vanish, for it remains on the characteristic curve. On the other hand, for a particle that moves with an arbitrary velocity u relative to the field of $p_{l,m,n}$,

$$\frac{dp_{l,m,n}}{dt} = \frac{\partial p_{l,m,n}}{\partial t} + u \frac{\partial p_{l,m,n}}{\partial x} \tag{66.3}$$

for the individual variation of $p_{l,m,n}$ is composed of the variation due to the change of the field and the variation due to the motion of the particle relative to the field. It follows, for the particle moving with the velocity c of the field, that

$$\frac{\partial p_{l,m,n}}{\partial t} + c \frac{\partial p_{l,m,n}}{\partial x} = 0 \tag{66.31}$$

Thus, the velocity of the characteristic curve

$$c = - \frac{p_{l,m,n+1}}{p_{l+1,m,n}} \tag{66.4}$$

the notation introduced by (66.1) being used.

In forecasting the future position of characteristic curves, it is important to know whether c will remain constant or not. To this end the variation of the velocity of the particle on the characteristic curve with the time dc/dt must be found. dc/dt is the acceleration A of the characteristic curve. If c is directed along the x-axis, the acceleration

$$A = \frac{dc}{dt} = \frac{\partial c}{\partial t} + c \frac{\partial c}{\partial x} \tag{66.5}$$

A can be found in the following manner: According to (66.3) the individual variation of $p_{l,m,n}$ of a particle moving with the velocity u relative to the field $p_{l,m,n}$ is obtained by applying the operator $\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$. Thus,

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} + 2u \frac{\partial^2}{\partial t \partial x} + \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x} + u^2 \frac{\partial^2}{\partial x^2}$$

For a particle remaining on the characteristic curve, $d^2p_{l,m,n}/dt^2$

must vanish. Because for such a particle $u = c$, it follows that

$$\frac{\partial^2 p_{l,m,n}}{\partial t^2} + 2c\frac{\partial^2 p_{l,m,n}}{\partial t\,\partial x} + \left(\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x}\right)\frac{\partial p_{l,m,n}}{\partial x} + c^2\frac{\partial^2 p_{l,m,n}}{\partial x^2} = 0 \tag{66.6}$$

The expression in the parenthesis of (66.6) is the acceleration. According to (66.5),

$$A = -\frac{p_{l,m,n+2} + 2cp_{l+1,m,n+1} + c^2 p_{l+2,m,n}}{p_{l+1,m,n}} \tag{66.7}$$

67. The Motion of Isobars and Isallobars. The equation for isobars has the form

$$p_{000} = \text{const}$$

Therefore, the velocity of isobars is

$$c_i = -\frac{p_{001}}{p_{100}} = -\frac{\partial p/\partial t}{\partial p/\partial x} \tag{67.1}$$

according to (66.4). The isobars move faster the greater the tendency and the smaller the pressure gradient. If the tendency is expressed by T and the distance between two consecutive isobars by h, (67.1) may be written

$$c_i = -Th \tag{67.11}$$

for h is inversely proportional to $\partial p/\partial x$.

The acceleration of the isobars

$$\begin{aligned} A_i &= -\frac{1}{\partial p/\partial x}\left(\frac{\partial^2 p}{\partial t^2} + 2c_i\frac{\partial^2 p}{\partial x\,\partial t} + c_i^2\frac{\partial^2 p}{\partial x^2}\right) \\ &= -\frac{p_{002} + 2c_i p_{101} + c_i^2 p_{200}}{p_{100}} \end{aligned} \tag{67.2}$$

according to (66.7). $\partial^2 p/\partial t^2$ can be determined either from two consecutive tendencies or, preferably, from three consecutive pressure readings at the same station, for

$$\frac{\partial^2 p}{\partial t^2} = (p_2 - p_1) - (p_1 - p_0) \tag{67.21}$$

provided that the time interval between the readings is chosen small enough. $\dfrac{\partial^2 p}{\partial x\,\partial t}$ is the gradient of the isallobars. It may be

expressed by $1/H$, the distance between two consecutive isallobars. $\partial^2 p/\partial x^2$ is the variation of the pressure gradient. Putting again

$$\frac{\partial p}{\partial x} = \frac{1}{h}$$

it follows that

$$\frac{\partial^2 p}{\partial x^2} = -\frac{1}{h^2}\frac{\partial h}{\partial x} \tag{67.22}$$

Thus, $\partial^2 p/\partial x^2$ is given by the variation of the distance between three consecutive isobars.

The equation for isallobars is of the form

$$p_{001} = \text{const}$$

The velocity of isallobars

$$c_T = -\frac{p_{002}}{p_{101}} = -\frac{\partial^2 p}{\partial t^2} H \tag{67.3}$$

according to (66.4). H is again the distance between two consecutive isallobars, and $\partial^2 p/\partial t^2$ can be found from (67.21). The acceleration of the isallobars can be determined in a similar manner, but in practice some of the necessary differential quotients of p cannot be determined with any degree of accuracy from the observations.

68. The Motion of Troughs, Wedges, and Pressure Centers. It will be assumed that there is no front in the trough so that the derivatives of the pressure are continuous. The motion of fronts will be discussed in the next section.

A trough line or a wedge line may be defined as a line along which the curvature of the isobars has a maximum value (Fig. 56*a* and *b*). The velocity of the line can be derived from this condition. However, it is possible to obtain a simplified but sufficiently accurate expression for the velocity of troughs and wedges from a less precise condition for the trough or wedge lines.

When the x-axis is tangential to the isobars at the intersection with the trough or wedge line,

$$\frac{\partial p}{\partial x} = 0 \tag{68.1}$$

at the trough or wedge, for the pressure along the x-axis must here have either a minimum or a maximum. The condition (68.1) is necessary but not sufficient for the existence of an extremum. Nevertheless, it yields a sufficiently accurate expression for the velocity of the trough or wedge line. From (66.4), it follows that the velocity of this line

$$c_L = -\frac{p_{101}}{p_{200}} \tag{68.2}$$

The variation of the pressure gradient in the x-direction, p_{200}, is negative for wedges and positive for troughs. A wedge moves,

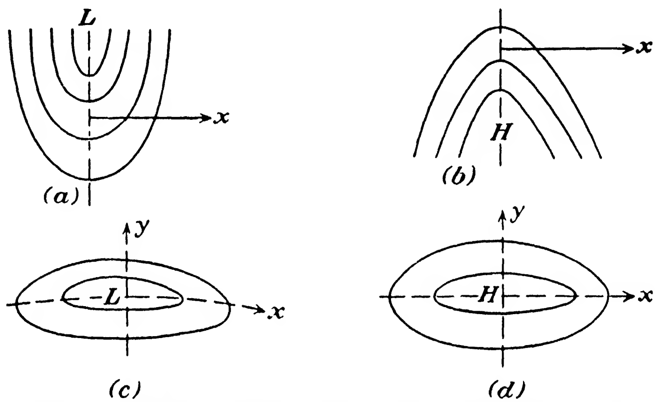

FIG. 56.—Trough (*a*), wedge lines (*b*), and pressure centers (*c*, *d*).

therefore, in the direction of greater positive tendencies and a trough in the direction of greater negative tendencies, as is, of course, obvious directly.

At a pressure center the pressure has a maximum or minimum relative to its surroundings. It is, in general, possible to draw two lines of symmetry through a pressure center, one connecting the points of maximum curvature of the isobars, the other connecting the points of minimum curvature (Fig. 56*c* and *d*). The definition of these symmetry lines corresponds to that of the trough lines (68.1). A pressure center can therefore as a rule be defined as the intersection between two lines having the characteristics of trough lines. The symmetry lines need not necessarily be at right angles to each other. Choosing one

of these lines as the x-axis and the other as the y-axis,

$$\frac{\partial p}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial y} = 0 \tag{68.3}$$

Let c_{Lx} and c_{Ly} denote the velocities of the two lines defined by (68.3). Then, in the same manner as for the trough and wedge lines,

$$c_{Lx} = -\frac{p_{101}}{p_{200}} \qquad c_{Ly} = -\frac{p_{011}}{p_{020}} \tag{68.4}$$

If the two symmetry lines are not at right angles, the future position of the center is found in the following manner: Upon multiplying c_{Lx} and c_{Ly} by the considered time interval Δt the displacements in the x- and y-direction are obtained. Because the displacement in the x-direction is $c_{Lx}\,\Delta t$, the pressure center O (Fig. 57) must be on the normal to the x-axis through A. It will as a rule not be on the x-axis unless the displacement in the y-direction has the right value. Owing to the latter displacement, O must be on the normal to the y-axis through B. It follows that the position of the pressure center after the time interval Δt is given by the intersection O' of both normals to the axes.

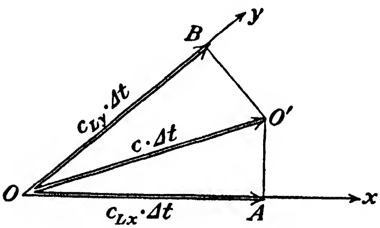

FIG. 57.—Velocity of a pressure center if the two symmetry lines are not at right angles.

The acceleration of troughs, wedges, and pressure centers requires the determination of higher derivatives of p which cannot be obtained with any great accuracy from the observations. Nevertheless, it is possible to derive from the acceleration formula (66.7) at least some qualitative rules for forecasting, as has been shown by Petterssen.

69. The Motion of Fronts. According to (60.1) the pressure must be the same on both sides of a surface of discontinuity. Consequently a front, being the intersection of a surface of discontinuity with the surface of the earth, may be characterized by the condition that

$$p^{\mathrm{I}} - p^{\mathrm{II}} = 0 \tag{69.1}$$

where p^{I} and p^{II} are the pressures on both sides of the front.

To find the velocity c_F of the front, $p^{\text{I}} - p^{\text{II}}$ may be substituted in (66.4) for $p_{l,m,n}$. It follows that[1]

$$c_F = -\frac{(\partial p^{\text{I}}/\partial t) - (\partial p^{\text{II}}/\partial t)}{(\partial p^{\text{I}}/\partial x) - (\partial p^{\text{II}}/\partial x)} \tag{69.2}$$

This formula is analogous to (67.1) for the velocity of an isobar except that now, instead of each derivative, the difference of the derivatives appears on both sides of the front.

Similarly, the acceleration of a front

$$A_F = -\frac{1}{\dfrac{\partial p^{\text{I}}}{\partial x} - \dfrac{\partial p^{\text{I}}}{\partial x}}\left[\frac{\partial^2 p^{\text{I}}}{\partial t^2} - \frac{\partial^2 p^{\text{II}}}{\partial t^2} + 2c_F\left(\frac{\partial^2 p^{\text{I}}}{\partial x\,\partial t} - \frac{\partial^2 p^{\text{II}}}{\partial x\,\partial t}\right) + c_F{}^2\left(\frac{\partial^2 p^{\text{I}}}{\partial x^2} - \frac{\partial^2 p^{\text{II}}}{\partial x^2}\right)\right]$$

according to (66.7).

70. The Application of the Kinematic Formulas to Forecasting. The formulas derived in the preceding sections of this chapter do not involve any dynamical principles. Therefore, as pointed out by Petterssen, they can not state anything about the causal relationships between the different atmospheric variations. They are extrapolation formulas that give the future position of characteristic curves. Even if the accelerations can be computed, the method employed is still an extrapolation method; but the reliability of this extrapolation will be improved when the accelerations can be determined, too.

Such extrapolation methods are, of course, always used tacitly in practical forecasting when conclusions with respect to the coming variations are drawn from the observed past variations, *e.g.*, when the future displacement of a cyclone is assumed to be similar to its past displacement.

Petterssen's method puts these extrapolations on a more satisfactory theoretical basis and permits the forecaster to take the structure of the pressure field into account when estimating the displacement and the variation of the pressure field.

[1] See also A. Gião, *Mém. Off. Nat. Mét. France*, No. 20, 41, 1929.

CHAPTER X

ATMOSPHERIC TURBULENCE

71. The Shearing Stresses in a Viscous Fluid. The viscosity produces a tangential force between fluid layers of different velocity. Consider a fluid originally at rest between two parallel plates at a distance D from each other (Fig. 58). When the upper plate is moved with a horizontal velocity U, while the lower plate remains at rest, the fluid in contact with the upper plate assumes also the velocity U due to friction, whereas the fluid in contact with the lower plate does not move.[1] Owing to the viscosity of the fluid produced by the irregular molecular motion the frictional drag exerted by the upper plate is transmitted to the lower plate, and it is necessary to apply a force to this plate in order to keep it at rest. Experiments have shown, as might be expected *a priori*, that the force exerted on the unit area of the stationary plate is proportional to the velocity U of the upper plate and inversely proportional to the distance D between the two plates. The ratio U/D is the velocity gradient perpendicular to the direction of motion, for experiments have shown that U decreases linearly from the moving to the stationary plate. The representation of the velocity distribution as shown in Fig. 58 is called the *velocity profile*, and we may say that in the case under consideration the velocity profile is linear.

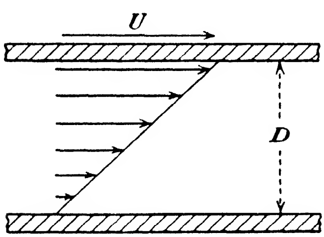

FIG. 58.—Frictional drag.

Analogously, the force τ per unit area exerted by a horizontal fluid layer with the velocity $u + du$ on a fluid layer with the velocity u at the distance dz is proportional to $\partial u/\partial z$. Introduc-

[1] This involves the assumption that there is no motion, relative to the solid boundary, of the fluid particles in direct contact with it.

ing a proportionality coefficient μ, we may write[1]

$$\tau = \mu \frac{\partial u}{\partial z} \tag{71.1}$$

τ is called the *shearing stress*, for $\partial u/\partial z$ is the shear of the velocity. The shearing stress has the dimension force/area (gm cm^{-1} sec^{-2} in cgs units) which is the same as the dimension of the pressure. μ is the *coefficient of viscosity*. Its numerical value for air at 760 mm Hg and 0°C is 1.71×10^{-4} gm cm^{-1} sec^{-1}. It increases with increasing temperature. The effect of the pressure on μ is quite negligible under ordinary conditions.

In the atmosphere, both the wind velocity and direction change with the elevation z. Therefore, the two velocity components u and v have to be considered separately. They give the components of the shearing stress

$$\tau_{zx} = \mu \frac{\partial u}{\partial z} \qquad \text{and} \qquad \tau_{zy} = \mu \frac{\partial v}{\partial z} \tag{71.2}$$

The velocity components u and v vary also in the x- and y-direction. However, in the study of atmospheric motions, the horizontal variations of u and v can frequently be neglected in comparison with the much larger vertical variations (see also Sec. 85). The stresses due to variations of the vertical motion can also be disregarded as a rule.

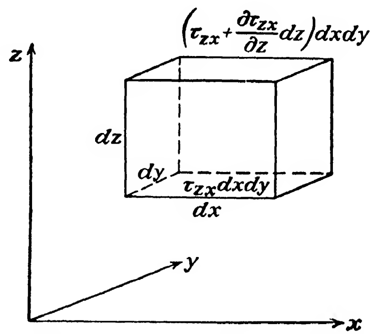

FIG. 59.—The force due to viscosity.

To make allowance for the effects of the shearing stresses in the hydrodynamic equations of motion (47.2) consider an infinitesimal parallelepiped (Fig. 59) with the edges dx, dy, and dz parallel to the coordinate axes. The drag exerted in the x-direction on the lower face $dx\,dy$ is $\tau_{zx}\,dx\,dy$; on the upper face $dx\,dy$ it is $\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\,dz\right] dx\,dy$. The difference between these two quantities gives the force acting on the volume element due

[1] This expression for the shearing stress was first introduced by Newton.

to the viscosity of the fluid,

$$\frac{\partial \tau_{zx}}{\partial z}\,dz\,dx\,dy = \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)dz\,dx\,dy$$

In the equations of motion, forces per unit of mass are considered. Therefore, the terms

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \tag{71.3}$$

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) \tag{71.4}$$

have to be introduced into the horizontal equations of horizontal motion which now become

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\omega\sin\varphi\, v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\omega\sin\varphi\, u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)\end{aligned} \tag{71.5}$$

These equations are derived under the assumptions that the vertical velocity component and the horizontal variations of the horizontal velocity components can be neglected. The complete equations of motion for a viscous fluid can be found in Lamb's "Hydrodynamics," Chap. XI.

72. Dynamic Similarity and Model Experiments. Although frequently the hydrodynamic equations can not be solved for a specific problem, it may be possible to study the problem on a model in the laboratory. The question arises then under which conditions the results obtained in the laboratory may be applied to the original problem. For such an application to be possible the boundaries of the flow in the model and in reality must be geometrically similar, and corresponding streamlines must be similar in shape. The latter condition requires that the various forces acting on corresponding fluid particles in the model and in reality are in the same ratio to each other.

We shall assume first that these forces are the pressure-gradient force, the frictional force, and the inertia force. Because they are in equilibrium, it is sufficient to consider only two of these

three forces, *e.g.*, the inertia force and the frictional force. The two systems, the model and the actual flow, may be characterized by two lengths l_1 and l_2—*e.g.*, the dimension of an obstacle or the width of a channel—and two corresponding velocities V_1 and V_2. The densities and viscosity coefficients may be ρ_1, μ_1 and ρ_2, μ_2, respectively. The inertia forces per unit volume are represented by expressions of the form $\rho \frac{\partial u}{\partial t}$ or $\rho u \frac{\partial u}{\partial x}$, so that the inertia forces can be expressed in the form $\rho V^2/l$ when the characteristic quantities of either system are used. Because the frictional terms are of the form $\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$, the characteristic expression for the frictional forces per unit volume is $\mu V/l^2$. Dynamic similarity requires that the ratio of the inertia forces to the frictional forces be the same at similar points in both systems. Thus the relation

$$\frac{\rho V^2}{l} : \mu \frac{V}{l^2} = \frac{\rho V l}{\mu} = \text{const} \tag{72.1}$$

is the condition for dynamic similarity. The dimensionless quantity $\rho Vl/\mu$ is called the *Reynolds number Re.*[1] If Re is large, the inertia forces are predominant; if it is small, the frictional force is predominant. The Reynolds number has a wide application in the study of turbulent flow.

If the effects not of viscosity but of gravity are considered, the postulate of dynamic similarity leads to another characteristic number. The force of gravity per unit volume is ρg (g is the acceleration of gravity). The ratio of inertia to gravity force is called *Froude's number Fr.* It follows that

$$Fr = \frac{V^2}{gl} \tag{72.2}$$

This number is widely used in model experiments for naval architecture, and it seems that it deserves similarly wide attention in meteorology.

Other factors that have to be taken into account in the study of dynamic similarity in the atmosphere are the earth's rotation

[1] Reynolds, O., *Phil. Trans.*, **174**, 935, 1883.

and the compressibility. Very general investigations including these factors are due to Helmholtz.[1] A discussion of meteorological model experiments has been given by Weickmann.[2]

Most of these experiments which aim at a demonstration of atmospheric currents on a small scale are only analogies to atmospheric motions, but other experiments may make real contributions to the solution of meteorological problems, such as the investigations of the air currents over obstacles, islands, or mountains[3] and around meteorological instruments[4] or the study of the motion of fluids injected into a rotating tank.[5]

73. Turbulent Motion. The velocity and direction of the fluid motion are frequently subjected to rapid irregular fluctuations. The motion is then called *turbulent.* When the motion of the fluid is made visible by the addition of a coloring substance, the lines of flow have a very irregular appearance, showing numerous small eddies, and cannot be followed for any considerable distance through the fluid. In the case of nonturbulent, or *laminar,* motion, on the other hand, the lines of flow have a smooth appearance and can be followed downstream from the point where the coloring substance was added, at least for a considerable distance.

The origin of turbulence has not yet been explained satisfactorily. Progress has been made in connection with the transition from laminar to turbulent flow, which begins when the Reynolds number increases over a certain critical value, and in the understanding of the effect of obstacles on the development of turbulence.

Atmospheric motion is always turbulent, as can be seen, for instance, from the inspection of the many fluctuations of a wind record or from the observation of the smoke from a chimney. The actual velocity u may therefore be split up into a *mean* velocity $\bar{u}$ and a *turbulent* velocity u' superimposed on the

[1] von Helmholtz, H., *Sitz.-Ber. Akad. Wiss. Berlin,* 501, 1873. See also F. M. Exner, "Dynamische Meteorologie," 2d ed., p. 92, Verlag Julius Springer, Vienna, 1925.

[2] In "Lehrbuch der Geophysik," by Gutenberg, B., Chap. 80, Gebrüder Bornträger, Berlin, 1929.

[3] Ahlborn, F., *Wiss. Ges. Luftfahrt,* No. 5, 1921.

[4] Bastamoff, S. L., and Witkiewich, W. J., *Bull. Geophys. Inst. Rech. Geophys.,* No. 10, 1926.

[5] Spilhaus, A. F., *J. Mar. Research,* **1,** 29, 1937.

mean velocity,

$$u = \bar{u} + u' \tag{73.1}$$

The mean velocity may be defined as the average of u for a time interval Δt which is sufficiently long to obtain a representative mean value that does not change too rapidly but, on the other hand, is not so long that the computation of the mean velocity obliterates meteorologically important changes of $\bar{u}$. Thus,

$$\bar{u} = \frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} u\, dt \tag{73.2}$$

It follows that for such a time interval the mean value of the turbulent velocity $\overline{u'}$ vanishes. The length of the suitable time interval Δt for the process of averaging may vary considerably under different conditions, but it appears that in many cases a time interval of 10 min will give representative values of the mean motion.

The hydrodynamic equations of motion (47.2) and continuity (47.3) are derived for the actual turbulent motion. Owing to its complexity, it is impossible to deal with this turbulent motion directly. The mean velocities defined above have to be introduced in the equations, and, in fact, these mean velocities are the ones which are of meteorological interest, and not the minor turbulent fluctuations. In introducing (73.1) and the corresponding relation for v and w in the hydrodynamic equations, additional terms due to the turbulent velocity components appear. It will be seen that these terms produce an effect analogous to that of the viscosity considered in the preceding section.

It is convenient to give the hydrodynamic equations a different form. The Coriolis terms may be omitted, for the reader will readily see how they are transformed. It will be assumed that the fluid is incompressible, but a similar transformation holds for compressible fluids. The equation for the x-direction may be written in the form

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

Upon multiplying the equation of continuity for an incompressible fluid by ρu,

$$\rho u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} + \rho u \frac{\partial w}{\partial z} = 0$$

and adding it to the preceding equation, it follows that

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = -\frac{\partial p}{\partial x} \tag{73.3}$$

Similar forms can be derived for the second and third equations of motion. The terms of Eqs. (73.3) may be integrated with respect to time for the interval from $t - \frac{\Delta t}{2}$ to $t + \frac{\Delta t}{2}$ in order to arrive at an equation for the mean motion. Now

$$\overline{uu} = \overline{(\bar{u} + u')^2} = \bar{u}\bar{u} + \overline{u'u'}$$

for $\overline{u'} = 0$. S milar expressions are obtained for $\overline{uv}$ and $\overline{uw}$. Thus, (73.3) becomes

$$\frac{\partial \rho \bar{u}}{\partial t} + \frac{\partial \rho \bar{u}\bar{u}}{\partial x} + \frac{\partial \rho \bar{u}\bar{v}}{\partial y} + \frac{\partial \rho \bar{u}\bar{w}}{\partial z} = -\frac{\partial p}{\partial x} - \frac{\partial \rho \overline{u'u'}}{\partial x} - \frac{\partial \rho \overline{u'v'}}{\partial y} - \frac{\partial \rho \overline{u'w'}}{\partial z} \tag{73.4}$$

Two analogous equations can be derived for the y- and z-components. The Coriolis terms would contain only the mean velocity but not the turbulent velocity, for they are of the first degree in the velocity terms. Likewise, the equation of continuity is of the first degree in the velocity terms and becomes, therefore,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{73.5}$$

Thus, the equation of continuity remains unchanged when the mean motion is considered except for the appearance of the mean velocity instead of the total velocity. In the equations of motion, on the other hand, additional terms appear that depend on the turbulent velocity. The quantities

$$\begin{matrix} -\rho\overline{u'u'}, & -\rho\overline{u'v'}, & -\rho\overline{u'w'} \\ -\rho\overline{v'v'}, & -\rho\overline{v'w'}, & -\rho\overline{w'w'} \end{matrix}$$

of which the latter three appear in the other two equations of motion are called the *eddy stresses.* They are due to the turbulent motion of the fluid which may be regarded as an eddying motion superimposed on the mean motion of the fluid. Their introduction into hydrodynamics is due to Osborne Reynolds. Equation (73.4) may be written again, with the aid of (73.5),

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{1}{\rho}\frac{\partial \rho\overline{u'u'}}{\partial x} - \frac{1}{\rho}\frac{\partial \rho\overline{u'v'}}{\partial y} - \frac{1}{\rho}\frac{\partial \rho\overline{u'w'}}{\partial z} \tag{73.6}$$

Thus, the equation of the mean motion contains terms due to the variation of the eddy stresses in space. It will be seen in the next section that these additional terms produce an effect similar to the effect of friction in a viscous fluid.

74. Prandtl's Theory of Momentum Transfer. In order to express the stresses in (73.6) by observable quantities, Prandtl[1] considers a fluid moving in the x-direction with the mean velocity $\bar{u}$. This mean velocity may be a function of the z-direction normal to the mean flow though it may be independent of the y-direction. Prandtl assumes, now, as a working hypothesis that a parcel of fluid which is displaced in the z-direction owing to the turbulent motion has to be moved a certain distance l before it loses its individuality by mixing with its new environment. The distance l is called the *mixing length.* It corresponds to the mean free path in the kinetic theory of gases although its physical definition is, of course, not so precise. It may be noted that the theory of the mixing length regards mixing as a discontinuous process which is obviously a working hypothesis only. When a parcel of fluid is displaced from a position z where the mean velocity is $\bar{u}\ (z)$ to $z + l$, the difference between its velocity and the velocity of the surroundings is $\bar{u}(z + l) - \bar{u}(z)$, or approximately $l\dfrac{\partial \bar{u}}{\partial z}$. This value is at least an approximation to the turbulent velocity component u'. To obtain the value of the z-component of the turbulent velocity w', consider two parcels of fluid coming from $z + l$ and $z - l$, respectively, to

[1] Prandtl, L., "Abriss der Strömungslehre," p. 93, F. Vieweg und Sohn, Braunschweig, 1931.

the level z. Their relative velocity will be $\pm 2l \dfrac{\partial \bar{u}}{\partial z}\cdot$ For reasons of continuity the turbulent vertical velocity component must therefore also be of the order of magnitude $l \dfrac{\partial \bar{u}}{\partial z}\cdot$ Thus,

$$\rho \overline{u'w'} \sim \rho l^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2$$

Prandtl writes an equality sign in this relation and thus includes the proportionality factor in l, which changes the meaning of l slightly. In order to show that the stress is positive if $\partial \bar{u}/\partial z > 0$ and negative if $\partial \bar{u}/\partial z < 0$, the preceding formula may be finally written

$$\tau_{zx}' = -\rho \overline{u'w'} = \rho l^2 \left|\frac{\partial \bar{u}}{\partial z}\right| \frac{\partial \bar{u}}{\partial z} \tag{74.1}$$

Similar expressions hold for the other eddy stresses. Those eddy stresses which are dependent not on the vertical but on the horizontal gradients of the mean velocity can in many instances be neglected in comparison with the eddy stresses due to the vertical variation of the horizontal mean velocities which are much larger (see pages 189 and 232).

The foregoing discussion shows that the turbulent state of the fluid produces a drag similar to the frictional drag. The formal analogy between the effects of viscosity and turbulence becomes more obvious by the introduction of a coefficient of eddy viscosity

$$\mu' = \rho l^2 \left|\frac{\partial \bar{u}}{\partial z}\right| \tag{74.2}$$

Because the effect of turbulence appears in the form of a frictional drag, it is customary to speak of the *eddy* viscosity of a fluid in turbulent motion and of the coefficient of eddy viscosity μ'; the ordinary viscosity of the fluid is referred to as the *internal* or *molecular* viscosity.

When the direction of the mean velocity changes with the altitude, the eddy stress

$$\tau_{zy}' = -\rho \overline{v'w'} = \rho l^2 \left|\frac{\partial \bar{v}}{\partial z}\right| \frac{\partial \bar{v}}{\partial z} \tag{74.3}$$

has to be considered, in addition to (74.1). Upon comparing (74.3) with (74.1), it will be noted that μ' need not necessarily be the same for τ_{zx}' and τ_{zy}', although for lack of information it is generally assumed that μ' is the same in both directions.

The hydrodynamic equations for horizontal mean motion in the turbulent atmosphere now obtain again the form (71.5) when the bars for the mean-velocity components and the prime distinguishing μ' and μ are omitted.

Taylor[1] has discussed the transformation of the eddy stresses under the assumption that not the momentum, but the vorticity of the motion is conserved if a parcel of fluid travels cross stream. With his assumption of a constant vorticity, it follows that the effect of eddy viscosity should be expressed by a term of the form $\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}$, not by $\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$ as in Prandtl's theory of momentum transport. A final decision in favor of one of these two forms cannot yet be made. When $\mu = \text{const}$, both expressions become identical. Ertel[2] has shown that both expressions are special forms of a more general form which follows from the tensorial nature of turbulence.

Prandtl's theory of the momentum transfer and Taylor's theory of the vorticity transfer treat turbulent mixing as a discontinuous process, an assumption that is obviously not very satisfactory. Taylor has therefore attacked the problem in an entirely different manner based on statistical methods.[3] He considers the coefficient of correlation r between the velocities of the fluid particles at a time t and at a later time $t + \Delta t$ and studies the dependence of r on Δt in the turbulent motion. His method was adopted and extended by Sutton.[4] For details of this statistical theory of turbulence the reader is referred to the original papers.[5]

75. The Vertical Variation of the Wind in the Surface Layer. Prandtl[6] has shown that a logarithmic law for the vertical

[1] Taylor, G. I., *Phil. Trans. Roy. Soc. A*, **215**, 1, 1915; *Proc. Roy. Soc. A*, **135**, 685, 1932.

[2] Ertel, H., *Ann. Hydr.*, **56**, 193, 1937.

[3] Taylor, G. I., *Proc. London Math. Soc.*, **20**, 196, 1922.

[4] Sutton, O. G., *Proc. Roy. Soc., A*, **135**, 143, 1932.

[5] See also D. Brunt, "Physical and Dynamical Meteorology," 2d ed., Chap. XII, Cambridge University Press, London, 1939.

[6] Prandtl, L., *Beitr. Phys. Atm.*, **19**, 188, 1932.

distribution of the wind velocity is obtained under the assumption that the surface layers of the atmosphere are subjected to the frictional drag at the ground and by the upper, faster-moving layers and that volume forces, such as the pressure-gradient force and the Coriolis force, can be neglected.

Under these conditions the shearing stress in the surface layers may be regarded as independent of the height. The mixing length may be assumed to increase at a linear rate with the distance from the earth's surface, which agrees with the laboratory experiments where the mixing length increases with the distance from the wall. The earth's surface has a certain roughness so that if $z = 0$ the mixing length has a finite value. Thus, according to Prandtl,

$$l = k_0(z + z_0) \tag{75.1}$$

k_0 is a nondimensional constant which according to Prandtl and von Kármán[1] has the value 0.38, approximately. z_0 depends on the roughness of the surface over which the air is flowing. It is called the *roughness parameter*.[2] Upon substituting (75.1) in (74.1), it follows that

$$\frac{\partial u}{\partial z} = \frac{1}{k_0(z + z_0)} \sqrt{\frac{\tau}{\rho}} \tag{75.2}$$

where the bars and primes have been omitted. Because ρ may also be regarded as a constant in the layers next to the earth's surface, the wind distribution follows a logarithmic law

$$u = \frac{1}{k_0} \sqrt{\frac{\tau}{\rho}} \ln \frac{z + z_0}{z_0} \tag{75.3}$$

The integration constant has been chosen so that $u = 0$ where $z = 0$.

The logarithmic law for the wind velocity in the lowest layers must be regarded as semiempirical, for the assumptions on which it is based are rather crude. Its main justification is its satisfactory agreement with the observations. Logarithmic laws for the wind distribution in the surface layer have, in fact, been suggested previously without any theoretical reasoning

[1] VON KÁRMÁN, Th., *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, p. 58, 1930.

[2] See also W. PAESCHKE, *Beitr. Phys. Atm.*, **24**, 63, 1937.

but simply from an inspection of the observations. Hellmann,[1] for instance, expressed his data by a formula of the form

$$u = a + b \log (z + c)$$

The following table shows some wind velocities observed by Hellmann and Köppen[2] (second line) and the wind velocities computed with the aid of the formula

$$u = 0.695 \ln \frac{z + 1.43 \text{ cm}}{1.43 \text{ cm}}$$

Height, cm	5	25	50	100	200	1600	3200
Wind velocity observed, m/sec	1.30	2.01	2.44	2.84	3.33	4.69	5.40
Wind velocity computed, m/sec	1.04	2.02	2.50	2.96	3.44	4.89	5.36

For the coefficient of eddy viscosity the following expression can be derived from (74.2) and the preceding equations:

$$\mu = \rho k_0(z + z_0) \sqrt{\frac{\tau}{\rho}} \tag{75.4}$$

Under the assumptions made by Prandtl the eddy viscosity is a linear function of the altitude in the surface layer. In order to express μ by observable quantities, Rossby and Montgomery[3] introduce the wind velocity u_a at the anemometer level z_a in (75.3). Upon substituting in (75.4), it follows that

$$\mu = \rho k_0^2(z + z_0) \frac{u_a}{\ln \dfrac{z_a + z_0}{z_0}} \tag{75.41}$$

The linear distribution of μ in the surface layer is confirmed by observations made by Mildner.[4]

The preceding theoretical considerations which lead to the logarithmic law of the wind distribution hold only in an atmosphere in indifferent equilibrium (Sec. 9) although a logarithmic formula appears also to approximate the observations well in

[1] Hellmann, G., *Met. Z.*, **32**, 1, 1915; **34**, 273, 1917.

[2] See A. Peppler, *Beitr. Phys. Atm.*, **9**, 114, 1921.

[3] Rossby, C.-G., and Montgomery, R. B., Papers in Physical Oceanography and Meteorology, Mass. Inst. Tech. and Woods Hole Ocean. Inst. **3**, 1935.

[4] Mildner, P., *Beitr. Phys. Atm.*, **19**, 151, 1932.

the case of a stable atmosphere, according to Sutton.[1] When the lapse rate is less than the adiabatic, the effect of the stability of the stratification has to be taken into account.

Richardson[2] and later Prandtl[3] showed that the turbulence in the atmosphere depends on the dimensionless quantity

$$\frac{\left(\frac{g}{\Theta}\right)\left(\frac{d\Theta}{dz}\right)}{\left(\frac{\partial u}{\partial z}\right)^2}$$

where Θ is the potential temperature. This expression is frequently referred to as the *Richardson number.* Its derivation is based on the reasoning that turbulence will die down when the work required to displace air from its equilibrium position in a stable atmosphere is greater than the work done by the eddy stresses. It will be seen on page 242 that an expression of the form $(\partial u/\partial z)^2$ is proportional to the work done by the eddy stresses. The numerical value for the Richardson number above which the motion ceases to be turbulent has not yet been determined definitely. Richardson found 1 and Prandtl 2 from theoretical considerations. Experiments of Prandtl suggest ¼ to ½ which would agree with investigations of Taylor[4] based on the study of small oscillations.

Rossby and Montgomery[5] have investigated the stabilizing influence of the stratification and found that with increasing stability the difference between the wind velocities at two heights near the surface tends to become proportional to the difference of the square roots at these two heights,

$$u_1 - u_2 \sim \sqrt{z_1} - \sqrt{z_2} \tag{75.5}$$

From observations as well as from theoretical considerations, laws of the form

$$u \sim z^{\frac{1}{n}} \tag{75.6}$$

[1] Sutton, O. G., *Quart. J. Roy. Met. Soc.*, **63**, 105, 1937.
[2] Richardson, L. F., *Proc. Roy. Soc. (London), A*, **67**, 354, 1920.
[3] Prandtl, L. *Beitr. Phys. Atm.*, **19**, 88, 1932.
[4] Taylor, G. I., *Proc. Roy. Soc. (London), A*, **132**, 499, 1931.
[5] Rossby and Montgomery *loc. cit.*

have been suggested for the wind distribution in the surface layers. The numerical values of n vary widely. In the homogeneous case a power law of the form (75.6) may be regarded as an approximation to the logarithmic law, for the logarithm is the limiting value for a very small positive power.

76. The Variation of the Wind above the Surface Layer. When the wind distribution above the surface layer is studied, the pressure-gradient force and the Coriolis force have to be taken into account. In order to simplify matters, it will be assumed that the pressure gradient is independent of the altitude, that the isobars are parallel straight lines, and that the motion is horizontal and steady.[1] The motion would then be geostrophic except for the influence of eddy and molecular viscosity. The coefficient of viscosity μ will be assumed constant, which is the simplest case, although in general it is known to vary with the altitude. The effects of the factors neglected here will be discussed in the following sections. When the x-axis is oriented parallel to the pressure gradient, the equations of motion become, according to (71.5),

$$\begin{aligned} -2\omega \sin \varphi \, v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{d^2 u}{dz^2} \\ 2\omega \sin \varphi \, u &= \frac{\mu}{\rho}\frac{d^2 v}{dz^2} \end{aligned} \tag{76.1}$$

Because u and v depend only on z, total differentials may be written instead of partial ones. The equation of continuity is obviously satisfied, for u and v are independent of x and y.

The simplest boundary condition at the ground would be the assumption that the wind velocity vanishes here. However, because, in the surface layer, equations different from (76.1) determine the wind distribution, it is preferable not to extend the solution down to the ground but only to a somewhat higher altitude, say up to anemometer level. It will be assumed that, at this level z_a, the shear of the wind is parallel to the wind itself, so that

$$u_a = \kappa \left(\frac{\partial u}{\partial z}\right)_a \quad \text{and} \quad v_a = \kappa \left(\frac{\partial v}{\partial z}\right)_a \tag{76.2}$$

[1] HESSELBERG, T., and SVERDRUP, H. U., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **1**, 241, 1915. ÅKERBLOM, F. A., *Nova Acta Reg. Soc. Sci. Upsal*, Ser. 4, **2**, 2, 1908.

The proportionality factor κ may be called the coefficient of surface friction, for it represents the effect of friction at the boundary separating the lowest surface layer from the layers that are now under consideration. If $\kappa = 0$, the boundary condition (76.2) is reduced to the simple condition that the velocity at the lower boundary of the layer vanishes.

An estimate of κ is possible using the results of the preceding section. It must be assumed that the wind is continuous at the boundary between the surface layer and the upper levels, for this boundary has no physical reality but is only a suitable surface above which another law for the wind distribution is chosen. This surface may, for instance, be placed at the anemometer level. Then

$$V_a = \kappa \left(\frac{dV}{dz}\right)_a \tag{76.21}$$

where V_a is the wind in the surface layer at this level. From (75.2) and (75.3), it follows that

$$\kappa = \frac{V_a}{(\partial V/\partial z)_a} = (z_a + z_0) \ln \frac{z_a + z_0}{z_0} \tag{76.22}$$

Here z_0 is the roughness parameter and z_a the anemometer level. If $z_0 = 4$ cm and $z_a = 10$ m, for instance, $\kappa = 55.2$ m.

In the following computations the height will be counted from the anemometer level so that $z_a = 0$. Introducing the geostrophic wind,

$$v_g = \frac{1}{2\omega \sin \varphi} \frac{1}{\rho} \frac{\partial p}{\partial x}$$

and

$$v' = v - v_g \tag{76.3}$$

Eqs. (76.1) may be written, because v_g is independent of z,

$$\begin{aligned} -2\omega \sin \varphi \, v' &= \frac{\mu}{\rho} \frac{d^2u}{dz^2} \\ 2\omega \sin \varphi \, u &= \frac{\mu}{\rho} \frac{d^2v'}{dz^2} \end{aligned} \tag{76.4}$$

This is a system of two homogeneous linear differential equations with constant coefficients. A solution of such a system must

consist of exponentials with real or complex exponents, in other words, of real exponential or trigonometric functions or a combination of both. Therefore, it appears justifiable to try whether expressions of the form[1]

$$\begin{aligned} u &= -Av_g e^{-az} \sin (az - b) \\ v' &= Bv_g e^{-az} \cos (az - b) \end{aligned} \tag{76.5}$$

satisfy Eqs. (76.4). Negative exponentials have been chosen, for u and v', which represent the deviations from the geostrophic wind, must tend to zero when the distance z from the ground and from its frictional drag tends to infinity. The choice of the negative exponentials in (76.5) thus represents the introduction of a second boundary condition, at infinity, in addition to the boundary condition (76.2), at anemometer level. The quantities A, B, and a and b are constants which have to be determined so that Eqs. (76.4) and the boundary conditions (76.2) are satisfied. The factor v_g has been added in order to make A and B dimensionless quantities. From (76.5), it follows that

$$\begin{aligned} \frac{du}{dz} &= Av_g ae^{-az}[\sin (az - b) - \cos (az - b)] \\ \frac{dv'}{dz} &= -Bv_g ae^{-az}[\cos (az - b) + \sin (az - b)] \end{aligned} \tag{76.51}$$

Upon differentiating again and substituting in (76.4), it follows that

$$-2\omega \sin \varphi \, v_g Be^{-az} \cos (az - b) = \frac{\mu}{\rho} 2a^2 Av_g e^{-az} \cos (az - b)$$

$$-2\omega \sin \varphi \, v_g Ae^{-az} \sin (az - b) = \frac{\mu}{\rho} a^2 2Bv_g e^{-az} \sin (az - b)$$

The exponential and trigonometric functions and the factor $2v_g$ can be canceled on both sides of these equations. Therefore,

$$\frac{B}{A} = -\frac{\mu a^2}{\rho\omega \sin \varphi} = -\frac{\rho\omega \sin \varphi}{\mu a^2}$$

[1] The reader who is sufficiently familiar with complex variables will have no difficulties in solving Eqs. (76.4) more directly by multiplying the second equation by $\sqrt{-1}$ and by determining the horizontal vector $u + \sqrt{-1}\, v$.

and

$$a = \sqrt{\frac{\rho\omega \sin \varphi}{\mu}} \tag{76.52}$$

Here the real positive root must be taken, for the exponential must tend to zero when $z \to \infty$. Furthermore,

$$B = -A$$

Thus

$$\begin{aligned} u &= -Av_g e^{-az} \sin (az - b) \\ v &= v_g[1 - Ae^{-az} \cos (az - b)] \end{aligned} \tag{76.521}$$

The remaining two constants A and b are found from the boundary conditions (76.2) at $z = z_a = 0$.

$$\begin{aligned} Av_g \sin b &= -\kappa A v_g a(\sin b + \cos b) \\ v_g(1 - A \cos b) &= \kappa A a\, v_g(\cos b - \sin b) \end{aligned} \tag{76.53}$$

From the first of these equations, it is found that

$$\tan b = -\frac{\kappa a}{1 + \kappa a} \tag{76.54}$$

and from the second, the value of $\tan b$ being used, that

$$A^2 = \frac{1}{1 + 2\kappa a + 2\kappa^2 a^2} \tag{76.55}$$

The constants A and b may also be expressed by the angle α_0 between the negative pressure gradient and the wind v_0 at anemometer level ($z = 0$). Because

$$\tan \alpha_0 = -\frac{v_0}{u_0} = -\frac{(\partial v/\partial z)_0}{(\partial u/\partial z)_0}$$

it follows from (76.51) that

$$\tan \alpha_0 = \frac{1 - \tan b}{1 + \tan b} \tag{76.56}$$

Upon substituting (76.54) the following relation is found between κ and α_0;

$$\tan \alpha_0 = 1 + 2\kappa a \tag{76.6}$$

When the wind at anemometer level vanishes, $\kappa = 0$, the angle between negative pressure gradient and wind is 45°. When $\kappa \neq 0$, α_0 is greater than 45°.

According to (76.56),

$$\tan \alpha_0 = \tan\left(\frac{\pi}{4} - b\right)$$

since $\tan \pi/4 = 1$. Therefore $b = (\pi/4) - \alpha_0$, or $b = (5/4)\pi - \alpha_0$. The second alternative would give a wind blowing toward higher pressure and has to be rejected. From (76.55) and (76.6), A may be expressed by α_0,

$$A^2 = 2\cos^2 \alpha_0 \tag{76.61}$$

Substituting the values of A and b in (76.521),

$$\begin{aligned} u &= -v_g\sqrt{2}\cos\alpha_0 e^{-az}\sin\left(az + \alpha_0 - \frac{\pi}{4}\right) \\ v &= v_g\left[1 - \cos\alpha_0\sqrt{2}e^{-az}\cos\left(az + \alpha_0 - \frac{\pi}{4}\right)\right] \end{aligned} \tag{76.7}$$

The wind direction is parallel to the geostrophic wind direction when $u = 0$. This direction is reached at the height where

$$az + \alpha_0 - \left(\frac{\pi}{4}\right) = n\pi \qquad (n = 1, 2, \cdots)$$

The lowest height D $(n = 1)$ at which this occurs is called the *gradient-wind level.* Thus

$$D = \sqrt{\frac{\mu}{\rho\omega\sin\varphi}}\left(\frac{5}{4}\pi - \alpha_0\right) \tag{76.71}$$

The gradient-wind level is higher, the larger μ and the smaller α_0. Further, it is inversely proportional to $\sqrt{\sin\varphi}$.

At the equator the gradient-wind level would become infinite, for here $\omega \sin \varphi = 0$ so that a geostrophic balance between frictional force, pressure-gradient force, and Coriolis force is impossible. The preceding formulas cannot be applied to the regions very close to the equator where no geostrophic balance is possible.

The gradient-wind level D and the angle α_0 which appear in (76.71) can be observed directly, although the determination of D may not yield very accurate values owing to the small deviation of the actual wind from the geostrophic wind near the gradient-wind level. It is therefore possible to compute μ

from observations, of course with the restricting assumptions (μ = const, v_g = const, steady straight-line flow) on which the solution (76.7) is based. Individual values of D and α_0 vary widely, depending on the weather situation and on the nature of the surface of the earth. To obtain an estimate of the order of magnitude of μ, it may be assumed that $\alpha_0 = 45°$ which corresponds to calm at $z = 0$ and $D = 1500$ m.[1] It follows,

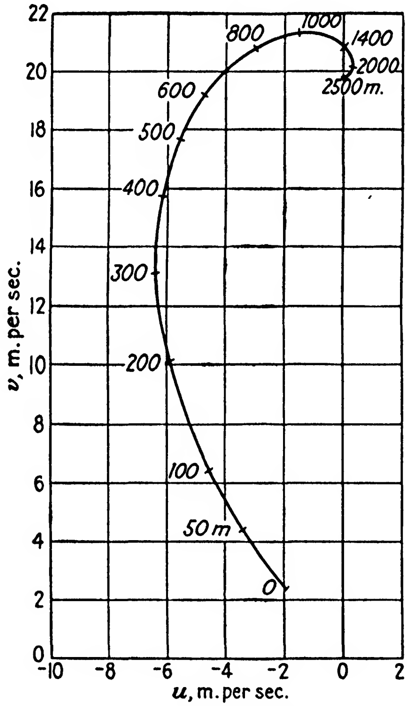

FIG. 60.—Vertical variation of the wind distribution (Ekman spiral).

when $\rho = 1.1 \times 10^{-3}$ gm/cm^3 and $\varphi = 40°$, that $\mu = 116$ gm cm^{-1} sec^{-1}. This value may be regarded, of course, only as a rough approximation owing to the simplifying assumptions and the varying atmospheric conditions. Even the order of magnitude may be smaller by a factor 1/10. But it is interesting to note that this value of μ is more than 10^5 times larger than the coefficient of molecular viscosity 1.71×10^{-4} gm cm^{-1} sec^{-1}.

[1] SEELIGER, W., *Beitr. Phys. Atm.*, **24**, 130, 1937.

Obviously, the value of μ determined from the height of the gradient-wind level is the coefficient of eddy viscosity, not the coefficient of molecular viscosity. The effect of the eddy viscosity in a turbulent medium such as the atmosphere is so much greater than the effect of the molecular viscosity that the latter can be neglected.

To give an example of the wind distribution represented by (76.7), let $\kappa = 55.2$ m, $v_g = 20$ m/sec, $\mu = 110$ gm cm^{-1} sec^{-1}, $\varphi = 40°$, and the average density $\rho = 1.1 \times 10^{-3}$ gm cm^{-3}. Then $a = 2.16 \times 10^{-3}$ m^{-1}, $\alpha_0 = 51°4'$, and $D = 1405$ m. The wind distribution computed from (76.7) is shown in Fig. 60. The curve represents the end points of the wind vector. The u-component can be read off the abscissa, and the v-component off the ordinate. The heights corresponding to a number of points are given on the curve. The end points of the wind vector lie on a spiral which is called the *Ekman spiral* after Ekman who solved the corresponding problem for ocean currents.[1] The first application to the meteorological problem was given by Akerblom.[2]

Figure 60 shows that the wind turns to the right (veers) with increasing altitude in the Northern Hemisphere. In the Southern Hemisphere the wind turns to the left, as can be seen from (76.7). Above the gradient-wind level the wind velocity becomes somewhat greater than the geostrophic wind, but the difference is small, for the exponential functions at and above the gradient-wind level are so small that the deviations from the geostrophic wind become rapidly negligible.

77. The Effect of the Vertical Variation of the Pressure Gradient. The expressions (76.7) were obtained under the assumption that the pressure gradient is independent of the altitude. If $-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ in (76.1) varies with the height, the equations represent a system of inhomogeneous equations that can be integrated by standard methods. In general, the height of the gradient-wind level is sufficiently small so that the pressure gradient may be regarded as constant up to the gradient-wind level. The rapid transition from the wind near the surface to

[1] Ekman, V. W., *Nyt. Mag. Naturv.*, **40**, 1, 1902.
[2] Åkerblom, *loc. cit.*

the gradient wind generally overshadows completely the small variation due to the change of the gradient wind in this layer.

Above the gradient-wind level, however, the effect of the variation of the pressure gradient with height may become of some importance. The effect of friction prevents the actual wind from attaining the gradient-wind values appropriate to a given altitude owing to the shearing stresses exerted by the upper and lower layers where the pressure gradient is different. Only the simple case will be dealt with here, that nothing but the intensity of the gradient wind changes. Upon introducing the geostrophic wind velocity v_g, Eqs. (76.1) may be written in the form

$$\begin{aligned} -2\omega \sin \varphi\, v &= -2\omega \sin \varphi\, v_g + \frac{\mu}{\rho} \frac{d^2 u}{dz^2} \\ 2\omega \sin \varphi\, u &= \frac{\mu}{\rho} \frac{d^2 v}{dz^2} \end{aligned} \tag{77.1}$$

According to the results of Sec. 76 the difference between v and v_g is very small above the gradient-wind level. Therefore, as pointed out by Mollwo,[1] v may be replaced by v_g in the second equation of (77.1) so that

$$u = \frac{1}{2\omega \sin \varphi} \frac{\mu}{\rho} \frac{d^2 v_g}{dz^2} \tag{77.2}$$

In order to obtain an estimate of the wind component normal to the isobars due to eddy viscosity above the gradient-wind level, let us assume that the vertical variation of v_g is caused by horizontal temperature gradients. According to (55.21)

$$\frac{\partial}{\partial z}\left(\frac{v_g}{T}\right) = \frac{g}{2\omega \sin \varphi} \frac{1}{T^2} \frac{\partial T}{\partial x}$$

For the sake of simplicity, T may be regarded as independent of z. Then

$$\frac{\partial^2 v_g}{\partial z^2} = \frac{g}{2\omega \sin \varphi} \frac{1}{T} \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x}\right)$$

According to Wagner[2] the meridional temperature gradient along the meridian 97° west between the latitudes 46° and 41.5°

[1] Mollwo, H., *Beitr. Phys. Atm.*, **22**, 45, 1935.

[2] Wagner, A., in Köppen-Geiger, "Handbuch der Klimatologie," Vol. 1, p. 7, Gebrüder Bornträger, Berlin, 1931.

north is 0.88°C per degree latitude at 500 m and 0.74°C/deg latitude at 4000 m, so that $\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x}\right) = 3.6 \times 10^{-14}$ °C/cm². From the above formula it follows that $\partial^2 v_g/\partial z^2 = 1.32 \times 10^{-9}$ per cm and sec if $T = 273°$ abs. Upon substituting in (77.2) and assuming that $\mu = 100$ gm cm^{-1} sec^{-1}, it is found that

$$u = 1.3 \text{ cm/sec.}$$

Although this is a rather small velocity, it may be of some importance in connection with pressure variations due to horizontal divergence and convergence. This possibility is discussed in some detail by Mollwo.

78. The Effect of the Centrifugal Force. It was assumed in Sec. 76 that the wind is geostrophic. The motion represents, then, a balance between pressure-gradient force, frictional force, and Coriolis force. In reality, however, the isobars and the paths of the air particles are frequently curved, and the centrifugal force has to be added to the three other forces. The mathematical problem to be solved becomes, in this case, more complicated, for the centrifugal force contains the square of the velocity so that the differential equation to be solved is of the second degree. A solution is possible under simplifying assumptions only.[1] In cyclones the gradient-wind level is reached at a lower altitude, in anticyclones at a higher altitude, than in the geostrophic case, when the pressure gradient, the coefficient of eddy viscosity, and the angle between surface wind and pressure gradient are the same in all three types of pressure distribution.

This result can easily be derived in a qualitative fashion. In cyclones the centrifugal force acts in the same direction as the Coriolis force. The centrifugal force is approximately v_{gr}^2/r, for the gradient-wind velocity v_{gr} may be substituted for the actual tangential velocity v. Thus, in the cyclonic case the angular velocity relative to the earth v_{gr}/r has to be added to the angular velocity of the earth's rotation $\omega \sin \varphi$. The parameter a [see (76.52)] in the expressions for u and v in Sec. 76 becomes

$$\sqrt{\frac{(v_{gr}/r) + \omega \sin \varphi}{\mu}\,\rho} \tag{78.1}$$

[1] Haurwitz, B., *Gerl. Beitr. Geophys.*, **45**, 243, 1935; **47**, 203, 206, 1936.

Upon substituting this term in the formula (76.71) for the height of the gradient-wind level D instead of $\sqrt{\frac{\rho\omega \sin \varphi}{\mu}}$, it is seen that D is smaller in the case of cyclonic motion than in the case of geostrophic motion.

In anticyclones, on the other hand, the centrifugal force acts in the direction opposite to the Coriolis force. Consequently, the expression corresponding to (78.1) is, for anticyclones,

$$\sqrt{\frac{\omega \sin \varphi - (v_{gr}/r)}{\mu}\rho} \tag{78.2}$$

which gives a higher gradient-wind level in anticyclones than in the geostrophic case, other things being equal. The end points of the wind vector are situated on a spiral similar to the one shown in Fig. 60.

79. The Variability of the Coefficient of Eddy Viscosity. It has been pointed out already that the coefficient of eddy viscosity must vary with the altitude. In the layer next to the ground, μ may be represented by a linearly increasing function of the height, according to Prandtl [see (75.41)]. At higher levels, μ must obviously follow a different law for it cannot increase indefinitely with elevation.

Methods for the direct determination of the vertical distribution of μ from the vertical wind distribution have been given by Solberg[1] and Fjeldstadt.[2] Solberg's method has been used by Mildner[3] and Schwandtke[4] to determine the distribution of μ. Fjeldstadt's method, which was originally developed for ocean currents, was extended to the atmosphere by Sverdrup.[5] All three investigations show clearly the variation of μ with the altitude. At first, μ increases from the ground upward; then it decreases again. Mildner found from pilot-balloon observations with two theodolites the following values of μ on Oct. 20, 1931, near Leipzig:

[1] In V. Bjerknes and Collaborators, "Physikalische Hydrodynamik," p. 502, Verlag Julius Springer, Berlin, 1933.

[2] Fjeldstadt, J. E., *Gerl. Beitr. Geophys.*, **23**, 237, 1929.

[3] Mildner, P., *Beitr. Phys. Atm.*, **19**, 151, 1932.

[4] Schwandtke, F., "Die innere Reibung der Atmosphäre in Abhängigkeit von der Luftmasse," dissertation, Leipzig, 1935.

[5] Sverdrup, H. U., Norwegian North Pole Expedition with the "Maud," 1918–1925, *Sci. Results*, Vol. 2, Met. I, p. 62, 1933.

z, m	80	135	190	240	295	405	460	510
μ, gm cm^{-1} sec^{-1}	125	270	310	500	246	117	70	70

The values of μ and their vertical distribution depend also on the character of the air mass in which the observations took place, especially on the vertical stability which is determined by the vertical lapse rate of temperature. In order to find a theoretical expression for the vertical distribution of μ, there must evidently be some additional conditions besides Eqs. (76.1). Rossby[1] obtains such conditions by an application of von Kármán's[2] postulate that the patterns of the turbulent-eddy disturbances are dynamically similar so that only the length scale and the velocity scale for each disturbance remain undetermined. In the case of atmospheric motions the Coriolis force has to be taken into account. Rossby showed that under these conditions von Kármán's principle requires that the absolute value of the shearing vector $\sqrt{(\partial u/\partial z)^2 + (\partial v/\partial z)^2}$ remain constant while its direction changes. It follows further that the shear of the wind and the rate of change of the shear must be at right angles to each other. The end points of the wind vector still form a spiral as in the simple case, when $\mu = \text{const}$, considered in Sec. 76; but the spiral is somewhat deformed.

For the mixing length in the layers above the surface layer, Rossby finds that

$$l = \frac{k(h - z)}{\sqrt{2}} \tag{79.1}$$

where k is a nondimensional constant whose value is estimated to be 0.065 and h is the height where the frictional influence disappears so that it corresponds to the gradient-wind level D [see (76.71)].

The coefficient of eddy viscosity

$$\mu = \rho l^2 \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}$$

by analogy with (74.2). Because it follows from Rossby's investi-

[1] Rossby, C.-G., *Mass. Inst. Tech. Met. Papers*, **1**, 4, 1932.

[2] von Kármán, T., *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, p. 58, 1938.

gations that the shear has the same value at every level in the layer under consideration,

$$\mu \sim (h - z)^2 \tag{79.2}$$

so that μ decreases above the surface layer. A similar distribution of μ was indicated by the figures obtained by Mildner (page 211) and by the results of other investigators. According to (79.2), μ should vanish at h, but the observations tend to show that even at the heights where the wind approaches the gradient wind there is still some residual turbulence. Rossby assumes that, for this residual turbulence, $\mu = 50$ gm cm^{-1} sec^{-1} under normal stable conditions. In unstable strata, μ may be 10 or even up to 100 times larger.

The extension of the principle of dynamic similarity to turbulent motion in the atmosphere is particularly valuable, for it permits one to derive not only the vertical wind distribution but also the vertical distribution of eddy viscosity. It is not necessary to start with a more or less arbitrary assumption concerning the dependence of μ on z, which is equally unsatisfactory from a theoretical and from a practical viewpoint.

In a later paper, Rossby and Montgomery[1] combined the solution found for the boundary layer next to the ground with the solution for the upper part of the frictional layer in which the wind distribution is represented by a spiral. The frictional drag, the mixing length, and the wind velocity must be continuous at the boundary between both parts of the frictional layer. From these conditions, it is possible to express the height of the boundary layer H by the height of the upper frictional layer. According to (75.1), $l = k_0 (H + z_0)$ at the height H of the boundary; according to (79.1), $l = kh/\sqrt{2}$, for, in the latter equation, z is counted from H upward. Thus

$$H + z_0 = \frac{k}{k_0 \sqrt{2}} h = 0.12h \tag{79.3}$$

This relation holds for an atmosphere in indifferent as well as in stable equilibrium.

[1] Rossby, C.-G., and Montgomery, R. B., "Papers in Physical Oceanography and Meteorology," **3**, 3, Mass. Institute of Technology and Woods Hole Ocean. Inst., Cambridge, Mass., 1935.

80. The Diurnal Variation of the Wind Velocity. At the earth's surface the wind velocity reaches normally—*i.e.*, apart from disturbances caused by the change in the weather situation—a maximum shortly after noon and a minimum during the early morning hours. At higher levels, the maximum of the wind velocity occurs during the night, the minimum about noon. Figure 61 shows the diurnal range of the wind velocity at various altitudes over Nauen and Potsdam according to observations by Hellmann.[1] Up to 32 m altitude, the maximum of the wind velocity is reached about noon, the minimum at night.

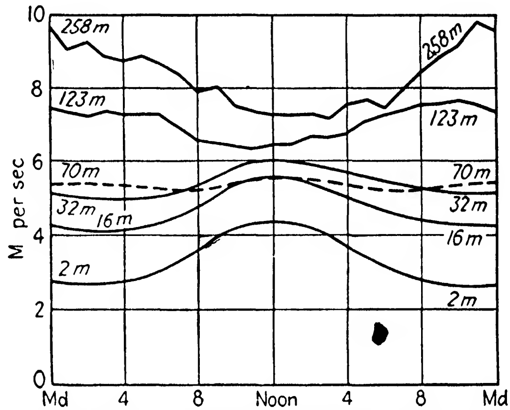

Fig. 61.—Diurnal variation of the wind velocity at different altitudes. (*After Hellmann.*)

At 123 m and above, the maximum occurs at night, the minimum in the daytime. The greater irregularity of the upper curves is due to the smaller number of observations. The anemometer at 70 m is obviously in the transition zone between the two types of wind variation. The curve for this level shows two faint minima, in the forenoon and afternoon, and two faint maxima, about midnight and noon. The data from which Fig. 61 is plotted represent mean values for the whole year. According to Hellmann's observations the daily variation of the wind velocity at 70 m is analogous to the variation at higher levels during the winter and to the variation at lower levels during the summer. Similar results have been obtained at other stations, for instance, on the Eiffel Tower.[2]

[1] Hellmann, G., *Met. Z.*, **34**, 273, 1917.
[2] Angot, A., *Ann. Bur. Centr. Mét. France*, p. 76, 1907.

A satisfactory theory of the diurnal variation of the wind velocity has been given by Wagner.[1] It is necessary to consider the equations of motion of a fluid (71.5) in which eddy viscosity is active,

$$\begin{aligned}\frac{du}{dt} - 2\omega \sin \varphi\, v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} + \frac{1}{\rho}\frac{\partial \mu}{\partial z}\frac{\partial u}{\partial z} \\ \frac{dv}{dt} + 2\omega \sin \varphi\, u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2} + \frac{1}{\rho}\frac{\partial \mu}{\partial z}\frac{\partial v}{\partial z}\end{aligned} \tag{80.1}$$

For simplicity, the acceleration terms have been written in undeveloped form, and the differentiations in the viscosity terms have been carried out to facilitate the following discussion. μ is, of course, to be regarded as the coefficient of eddy viscosity.

According to the observations,

$$\frac{\partial u}{\partial z} > 0, \qquad \frac{\partial v}{\partial z} > 0, \qquad \frac{\partial^2 u}{\partial z^2} < 0, \qquad \frac{\partial^2 v}{\partial z^2} < 0$$

in the layer that is here considered. Therefore, when $\partial\mu/\partial z > 0$, the last terms in both equations (80.1) cause an increase of the wind velocity with the time, whereas the terms with μ cause a decrease of the wind velocity. In the layers next to the ground, $\partial\mu/\partial z$ is positive according to (75.4). The value of $\partial\mu/\partial z$ attains a maximum around noon in the layers next to the ground, owing to the effect of the surface heating and the resulting vertical convection. Because μ itself is always very small at the ground, the last term in each of the two equations (80.1) is more effective than the preceding one. Thus, the noon maximum of the wind velocity near the ground is due to the increase of $\partial\mu/\partial z$ and occurs *in spite* of the increase of μ in the lowest layers. An increase of μ alone would cause a decrease of the wind velocity. The effect of the increase of μ during the noon hours is predominant, however, in higher layers, so that here the wind velocity sinks to a minimum about noon. At night the wind velocity has a maximum in the upper layers because μ is smaller than in the daytime and a minimum in the layers next to the ground owing to the decrease of $\partial\mu/\partial z$ from day to night.

The diurnal variation of the wind velocity in the layer next to the ground is also affected by the friction at the ground.

[1] Wagner, A., *Gerl. Beitr. Geophys.*, **47**, 172, 1936.

The stratification of the surface layers is less stable in the daytime, more stable at night. Thus, the air at the surface can surmount obstacles at the surface more easily in the daytime than at night when the more stable stratification requires a greater expenditure of energy. Consequently, the surface friction is smallest during the noon hours and greatest during the night; this factor also contributes to the maximum of the surface wind around noon and to the minimum during the night.

The wind direction has also a diurnal period which is, however, more disturbed by local effects, such as land and sea breezes or mountain and valley winds, than the diurnal period of the wind velocity. Wagner has shown that this periodic variation of the wind direction is also due to the daily variation of μ and $\partial\mu/\partial z$.

Problems

22. Find the relation between pressure gradient and wind velocity and the angle between pressure gradient and wind if a frictional force acts that is proportional to the wind velocity and in the direction opposite to the wind. Assume geostrophic conditions apart from the effects of friction.

23. Derive a relation between the direction and velocity of the surface (anemometer-level) wind and the geostrophic wind velocity under the assumptions made in Sec. 76.

24. Introducing the assumptions made in Sec. 76, find the angle between the frictional force and the pressure gradient. What is the angle between the direction of the wind and the frictional force at the surface (anemometer-level) and at gradient-wind level?

25. Show that the wind component due to friction above the gradient-wind level is zero if the temperature gradient is in the same direction as the pressure gradient and if the temperature decreases linearly with the altitude but at the same rate everywhere.

CHAPTER XI

TURBULENT MASS EXCHANGE

81. Transfer of Air Properties by Turbulent Mass Exchange. The transfer of momentum at right angles to the direction of mean motion has already been discussed in Sec. 74. We have seen that the turbulent eddying motion produces an effect similar to the molecular viscosity, only of much greater intensity, as is shown by the fact that the coefficient of eddy viscosity is 10^5 times larger than the coefficient of molecular viscosity (see page 206).

In a similar manner the turbulent eddies cause also a transfer of heat, of water vapor, and of other properties of the air. The heat transfer is analogous to the molecular conduction of heat; the transfer of water vapor is analogous to the process of molecular diffusion. But the effectiveness of the transfer by turbulent eddies is much greater than the effectiveness of the molecular transfer, as can be expected from the comparison of the coefficients of eddy viscosity and molecular viscosity. Because the transfer of properties by the turbulent eddies involves the transport of air, it is frequently referred to as *turbulent mass exchange.* Physically the transfer of momentum considered in the last chapter and the transfer of other properties to be discussed in this chapter are entirely analogous. The special case of heat transfer by turbulent mass exchange is sometimes also referred to as *eddy conductivity.*

The amount of the property per unit mass of air whose transfer is to be studied may be denoted by s. The quantity s will be assumed to vary in the vertical direction only. This assumption is in many cases justified, for the vertical variations of the air properties are in general much larger than the horizontal ones. The following discussion can, moreover, easily be extended to three dimensions.

The quantity s may represent any property of the air that does not change owing to vertical motion. Thus potential temperature or mixing ratio may be considered, as long as no

condensation takes place, but not the temperature or the relative humidity.

Consider a horizontal area F at a height z that is moving with the mean horizontal-wind velocity. It will be assumed that there is no *mean* vertical component of the air motion. This assumption must be fulfilled, at least on the average, to satisfy the continuity condition. Owing to the turbulent eddying motion a vertical transport of air upward and downward through F is taking place. Upon denoting the mass of a parcel of air ascending through F by m_+ and the mass of a particle descending through F by m_-, the total amount of air rising through F must be equal to the total amount of air descending through F,

$$\sum_F m_+ = \sum_F m_- \tag{81.1}$$

We shall suppose now as in Sec. 74 that a particle travels a certain distance l, the mixing length, before mixing with its surroundings.[1] The distance l is not necessarily the same for all particles passing through F. The amount of s carried by an air particle of mass m arriving at F will be $m\,s(z \pm l)$ where the argument of s indicates that the value of s is to be taken at the level $z \pm l$. The positive sign has to be chosen if the particle descends, and the negative sign when it ascends. The average time t may be needed by the particle to move through the distance l to F. Then, the net transport of s upward per unit time and unit area

$$S = \frac{1}{Ft}\left[\sum_F m_+\, s(z - l) - \sum_F m_-\, s(z + l)\right] \tag{81.2}$$

Because l is a small quantity,

$$s(z \pm l) = s(z) \pm l\left(\frac{ds}{dz}\right)_z + \frac{l^2}{2}\left(\frac{d^2s}{dz^2}\right)_z + \cdots \tag{81.3}$$

Here terms of higher than the second order in l may be omitted. In the neighbourhood of z, s is thus expressed by the value of s at z and its derivatives. Upon substituting (81.3) in (81.2) and noting that s and its derivatives *at the level* z can be taken before

[1] See also W. Schmidt, "Der Massenaustausch in freier Luft und verwandte Erscheinungen," Henri Grand, Hamburg, 1925. The above analysis follows Schmidt mainly.

the summation sign, it follows that

$$S = \frac{1}{Ft}\left[s(z)\left(\sum_F m_+ - \sum_F m_-\right) - \frac{ds}{dz}\left(\sum_F m_+ l + \sum_F m_- l\right) + \frac{1}{2}\frac{d^2s}{dz^2}\left(\sum_F m_+ l^2 - \sum_F m_- l^2\right)\right]$$

The suffixes z added to ds/dz and d^2s/dz^2 in (81.3) have been omitted again, for in the following these quantities are always to be taken at the level for which the transport is to be computed. The first term on the right in the equation above vanishes on account of (81.1). Similarly, the third term vanishes because, for each mass coming from a distance l above F, there will be, on the average, a mass coming from the same distance below F, except very close to the ground. The factor of $\frac{ds}{dz}$ may be written Σml where the summation is now to be extended over all positive and negative m. Thus

$$S = -\frac{\Sigma ml}{Ft}\frac{\partial s}{\partial z} = -A\frac{\partial s}{\partial z} \tag{81.4}$$

where

$$A = \frac{\Sigma ml}{Ft} \tag{81.5}$$

The coefficient A is the coefficient of turbulent mass exchange. Its dimensions are in grams per centimeter per second when cgs units are used, the same as the dimensions of the coefficient of eddy viscosity with which it is, in fact, identical when the exchange of momentum is considered.

In some cases, it is necessary to add another factor to Eq. (81.4) in order to have the same physical dimensions on both sides of the equation. For instance, when the heat transport in relation to the gradient of potential temperature is considered, the right-hand side of (81.4) has to be multiplied by the specific heat at constant pressure c_p. The heat transport due to turbulent mass exchange is thus

$$S = -c_pA\frac{\partial \Theta}{\partial z} \tag{81.41}$$

A derivation of the transfer by turbulent mass exchange differing from Schmidt's has been given by Taylor,[1] who showed that

$$A = \overline{\rho w' l} \tag{81.6}$$

where ρ is the density of the air, w' the turbulent vertical motion, and l the mixing length, as before. The bar indicates that the mean value of the product should be taken. It will be noted that Taylor's expression for A is very similar to Schmidt's.

The coefficient A is, of course, a function of the altitude and of the time because it depends on the intensity of the turbulent mixing. When the stability of the atmospheric stratification is great, as, for instance, at inversions, the turbulent mass exchange is very small. Around noon, especially during times of strong insolation, on the other hand, A will be large.

To show the strong variation of A with height next to the ground Schmidt[2] discusses some temperature measurements made by Wüst in the lowest 9 m over the Baltic Sea. If the transport of heat had not been the same at all levels, some layers would have gained or lost heat and thus their temperature would have changed. But the temperature remained constant at all levels up to 9 m, and therefore the heat transport must have been the same everywhere. Thus, it follows from (81.4) that for two heights

$$A_1 : A_2 = \left(\frac{ds}{dz}\right)_2 : \left(\frac{ds}{dz}\right)_1$$

For s the temperature may be taken here instead of the potential temperature, for the heights considered are so small that adiabatic changes of the temperature at the rate of 1°C/100 m are of no effect. The temperature gradient at various levels and the ratio of A at each level to the value at 1 m are given in the following table:

Height, m	0.2	0.5	1.2	2.0	4.0	6.0	9.0
dT/dz, deg C/m	326	42	14.0	8.6	6.4	3.3	1.3
$A : A$ (1 m)	0.05	0.4	1.2	2.0	2.6	5.1	12.9

[1] TAYLOR, G. I., *Phil. Trans. Roy. Soc. A*, **215**, 1, 1915.
[2] SCHMIDT, *loc. cit.*

The coefficient of turbulent mass exchange is very small near the ground where vertical motions must be small and increases very quickly upward.

Besides the potential temperature and the mixing ratio the preceding considerations may also be applied to such properties of the air as its dust content. In this case, allowance must be made for gradual settling of the dust. An application to the entropy is, however, not permissible; for the entropy, though remaining constant during adiabatic motion, changes during mixing processes, as pointed out by Pekeris.[1]

It is obviously not *a priori* to be expected that the coefficient of turbulent mass exchange is the same for all properties transferred by mixing. The mixing length l may be different for different quantities. From observations of Sverdrup,[2] it appears, however, that in the atmosphere A is the same for momentum and heat whereas in the ocean it is different.

82. The Differential Equation of Turbulent Mass Exchange. The transfer S of a property s is a function of the height. Thus, the transfer of s per unit area and unit time across the level z is $S(z)$, and the transfer across the level $z + dz$

$$S(z + dz) = S(z) + \frac{\partial S}{\partial z}\, dz$$

when dz is infinitesimally small. A rectangular parallelepiped of unit cross section and of the height dz gains, therefore, per unit time the amount

$$S(z) - \left[S(z) + \frac{\partial S}{\partial z}\, dz\right] = -\frac{\partial S}{\partial z}\, dz$$

The amount of mass in the parallelepiped is $\rho\, dz$. Thus, the total change of s per unit time inside the parallelepiped is $\rho \frac{\partial s}{\partial t}\, dz$. Upon equating both expressions for the change of s inside the parallelepiped, it follows that

$$\rho \frac{\partial s}{\partial t}\, dz = -\frac{\partial S}{\partial z}\, dz$$

[1] Pekeris, C. L., *Met. Z.*, **47**, 231, 1930.

[2] Sverdrup, H. U., *Geofys. Pub.*, **11**, No. 7, 1936.

or, with (81.4), that

$$\frac{\partial s}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(A\,\frac{\partial s}{\partial z}\right) \tag{82.1}$$

This is the differential equation of turbulent mass exchange. It has the same form as the equation of molecular heat conduction, and therefore the solutions derived in the theory of heat conduction can also be applied to the analogous problems arising in the discussion of the effect of turbulent mass exchange. If A is assumed to be constant, (82.1) can be written in the form

$$\frac{\partial s}{\partial t} = \frac{A}{\rho}\frac{\partial^2 s}{\partial z^2} \tag{82.2}$$

When A is given, the vertical distribution of s and its derivatives with time can be computed. Frequently, however, the opposite problem arises, that s is given and that A is to be found.

Ertel[1] has shown that A can be determined by means of the standard deviations of s and $\partial s/\partial t$ which are available from records of the turbulent variations. This method has been used by Lettau[2] and others to find A from wind and temperature records.

Other more indirect methods of determining A can be developed by integration of (82.1) under suitable assumptions about the dependence of A on z and t. Upon applying the solution of the differential equation to the observations of s, it is possible to determine the value of A. As an example, the daily temperature period will be considered in the next section. The main difficulty of this procedure is that the dependence of A on z and t is not definitely known, and thus only a rough determination of A is possible.

83. The Daily Temperature Period. The air temperature near the earth's surface undergoes a periodic daily change, with a maximum around two hours past noon and a minimum around sunrise. On clear days the amplitude of the daily temperature period is greater than on cloudy days. At greater heights the amplitude is smaller, and the maximum occurs later than at the surface. The decrease of the amplitude with the elevation

[1] ERTEL, H., *Gerl. Beitr. Geophys.*, **25**, 279, 1930.

[2] LETTAU, H., *Ann. Hydr.*, **62**, 469, 1934; "Atmosphärische Turbulenz," Akademische Verlagsgesellschaft, Leipzig, 1939.

and the retardation of the maximum are due to the propagation of the temperature wave upward by turbulent mass exchange, although other influences are effective, too, as will be seen from the following discussion.

The daily temperature period at the ground depends on the rate of heating, mainly by direct solar radiation, in the daytime and on the rate of cooling, mainly by nocturnal radiation (see Sec. 39), at night. In the daytime the temperature curve follows the variations of the intensity of the solar radiation quite closely and can therefore be approximated by a cosine function. At night the temperature decreases gradually to a minimum in the morning hours around sunrise. Therefore, the temperature variation throughout the whole day cannot be represented exactly by a single cosine function, although it can be expressed accurately enough by a few terms of a harmonic series. Nevertheless, for the sake of convenience, we shall express the daily temperature variation here by a single cosine term. The following computations can easily be extended to a trigonometric series. Let the temperature at the surface

$$T_0 = B \cos \nu t \tag{83.1}$$

Here the daily mean temperature at the surface has been given the value zero, B is the amplitude, and $\nu = 2\pi/\text{day}$. The phase angle has been put equal to zero so that the time t is counted from the time of the temperature maximum at the ground. In order to apply (82.1) to the problem, the potential temperature should be considered according to page 216. However, we are interested only in the comparatively small periodic deviations of the temperature from the mean values at each altitude and not in the mean vertical temperature distribution; therefore, the temperature may be used instead of the potential temperature. For the sake of simplicity, it will be assumed that $A = \text{const}$. The differential equation to be solved is then, according to (82.2),

$$\frac{\partial T}{\partial t} = \frac{A}{\rho} \frac{\partial^2 T}{\partial z^2} \tag{83.2}$$

subject to the condition (83.1) when $z = 0$. The observations indicate that the amplitude whose value at the earth's surface is B decreases with the altitude and that the maximum occurs

later at higher levels. As a function that satisfies these conditions and that is equal to (83.1) when $z = 0$, we may choose

$$T = Be^{-\lambda z} \cos (\nu t - \lambda z) \tag{83.3}$$

Here the decrease of the amplitude is expressed by $e^{-\lambda z}$, and the retardation of the time of the temperature maximum by λz. Upon substituting (83.3) in (83.2), it follows that this expression is a solution of the differential equation provided that

$$\lambda = \sqrt{\frac{\nu \rho}{2A}} \tag{83.4}$$

When the amplitudes r_1 and r_2 at the levels z_1 and z_2 are observed,

$$\frac{r_1}{r_2} = \frac{Be^{-\lambda z_1}}{Be^{-\lambda z_2}}$$

or

$$\lambda = \frac{1}{z_2 - z_1} \ln \frac{r_1}{r_2} \tag{83.41}$$

When the phase retardation of the maximum $\Delta \chi$ between the levels z_1 and z_2 is known, it follows that

$$\lambda = \frac{\Delta \chi}{z_2 - z_1} \tag{83.42}$$

The coefficient of turbulent mass exchange can thus be computed from the vertical distribution of the amplitude or of the phase. If the variation of the mean temperature in the vertical direction is to be included in (83.3), the differential equation (83.2) may be regarded as an equation for the potential temperature Θ. Obviously, any linear function of z, *e.g.*,

$$\Theta = \Theta_0 + \beta z$$

satisfies the differential equation (83.2). From (9.31), it follows that

$$T = T_0 - (\Gamma - \beta) z$$

for at the earth's surface, very closely, $T_0 = \Theta_0$. In order to have a more complete solution the linear expression for T may be added to (83.3). Because (83.2) is a linear differential equation, the sum of two solutions is also a solution. But,

as mentioned previously, for the present discussion of the daily temperature period only the periodic term is of importance.

Equation (83.3) together with (83.4) shows that, the larger ν, *i.e.*, the smaller the period $2\pi/\nu$, the more rapidly does the amplitude of the period decrease. On the other hand, the greater A, the less rapidly does the amplitude diminish. The smaller the period $2\pi/\nu$, the larger the phase angle λz which shows the retardation of the maximum as a function of the period. Furthermore, the larger A, the smaller the phase angle. It will be noted that similar conditions hold for the propagation of a temperature wave by molecular conduction.

A convenient representation of the vertical distribution of the temperature period can be given in a polar diagram as shown by Schmidt.[1] The amplitude r and the phase angle χ are represented, respectively, by the length of the radius vector and by the angle that it includes with an arbitrary zero direction. In the present case of constant A,

$$r = Be^{-\lambda z}$$

and

$$\chi = \lambda z$$

so that

$$r = Be^{-\chi}$$

The end points of the radius vector should therefore be situated on a logarithmic spiral around the origin. Schmidt uses temperature observations made on the Eiffel Tower during 5 years. The following table shows, as an example, the average amplitude and time of maximum during autumn at different heights:

Height, m	Amplitude, deg C	Time of maximum, P.M.
1.8	3.00	2.5^h
123.1	2.09	4.0^h
196.7	1.72	4.5^h
301.8	1.29	4.5^h

These data are plotted in the polar diagram in Fig. 62. The points lie on a spiral. The spiral, however, approaches, not

[1] SCHMIDT, *op. cit.*, p. 21.

the origin, but a point R to which correspond the amplitude 1.23°C and the time of maximum temperature 3.5^h P.M. Schmidt ascribes the discrepancy between the theory and the observations to the effect of direct heating of the air by solar radiation. But the radiation cannot be the sole reason. In reality, A varies with time and altitude contrary to the assumption of a constant A, and thus the points cannot be expected to lie on a spiral around O.

Choosing R in Fig. 62 as the new origin, Schmidt obtains new values for the amplitudes and for the retardation of the maximum with elevation. From the latter, he finds as the value of A

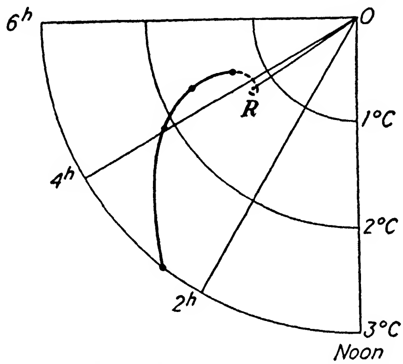

FIG. 62.—Polar diagram of the daily temperature period on the Eiffel Tower. (*After W. Schmidt.*)

9 gm cm^{-1} sec^{-1} for the layer between the two lowest levels and 15 and 11 gm cm^{-1} sec^{-1} for the next two layers. These figures indicate clearly that A varies with the altitude, but it is impossible to find reliable values for this *variation* of A based on a solution that starts from the assumption that A = const. Even the order of magnitude of the mean value for A throughout the whole layer may be completely wrong when A is computed from the observations under the assumption that it is constant whereas it is in reality variable with altitude. The simplifying assumption that A = const gives only a qualitative picture of the decrease and retardation of the temperature wave with the altitude.

Haurwitz[1] has studied the vertical distribution of the amplitude and phase retardation under the assumption that A increases

[1] HAURWITZ, B., *Trans. Roy. Soc. Canada*, 3d ser., Sec. III, **30**, 1, 1936.

linearly with height. Although this assumption is also arbitrary, it has at least the advantage that the determination of the variability of A with z need not be based on a solution which supposes that A = const. to begin with. Moreover, for the lowest layers, A may be expected to be a linear function of z, by analogy with the results of Rossby and Montgomery for the coefficient of eddy viscosity (*cf.* page 212) which is closely related to the coefficient of turbulent mass exchange.

When A increases with the altitude, the amplitude of the daily temperature variation decreases rapidly in the lowest layers, while higher up the decrease is slower. Similarly, the phase retardation increases most strongly in the lowest layers while it is constant in the case of constant A. The rapid variation of amplitude and phase in the lowest layers is due to the very small A near the ground. A small A causes a rapid decrease of the amplitude and a rapid increase of the phase retardation as can be seen from Eqs. (83.3) and (83.4) for constant A. The paper describes also how the vertical distribution of A can be computed from three observations of the amplitude or of the phase under the assumption that A increases linearly with the height. Upon applying this method to the Eiffel Tower observations, values are obtained that are noticeably higher than those given by Schmidt. The latter values appear too small when compared with results from other observations, also.

The coefficient of turbulent mass exchange not only depends on the altitude but also varies with time. In the daytime, especially around noon when strong convection occurs, A is larger than at night when the stratification of the atmosphere is stable. The effect of the variation of A with time makes the determination of A from the daily temperature period still less reliable, although it is possible to solve (83.2) also without difficulty if A/ρ is a periodic function of the time.

84. The Transformation of Air Masses by Turbulent Mass Exchange. In Sec. 39, we discussed the transformation of a relatively warm maritime air mass that comes to rest over a cold, snow-covered continent. The cooling of such an air mass must largely be due to radiative processes. In addition to the radiative transfer of heat, there must also be cooling due to the transfer of heat by turbulent mass exchange from the air to the ground. But in the case under consideration the turbulent mass

exchange must be small owing to the very stable stratification of the air.

On the other hand, when a relatively cold air mass comes in contact with a warmer surface, it will be heated from the surface upward owing to the effect of turbulent mass exchange. This effect will be much stronger now than in the case of a warm air mass over colder ground, for the stability of the stratification is much smaller.

In this section, we shall study the transport of heat upward in somewhat greater detail. It is hardly necessary to mention that the transfer of other properties, *e.g.*, of specific humidity, may be discussed in the same fashion.

We have to consider now the potential temperature Θ. Once Θ is known, the temperature T can easily be obtained from Sec. 9: The potential temperature Θ may be a linear function of the altitude in the cold air, before the motion toward warmer regions has taken place. If the time is counted from the moment when the air reaches the warmer region,

$$\Theta = \Theta_0 + \gamma z \qquad \text{when} \qquad t = 0 \tag{84.1}$$

This assumption is justified when inversions such as are shown in Fig. 23 for polar continental air are not present. It would also be possible to solve the problem for more complicated distributions of Θ, but the solution is more laborious. As a further simplification, the transition of the air from the colder to the warmer region may be sudden. This is never strictly true even during the motion from land to sea. In fact, when a cold mass moves in a southerly direction over land or over water, the temperature variation is rather gradual. The case of a gradual variation of the surface temperature can be treated quite easily,[1] but here we shall restrict ourselves to the simpler condition that when $t > 0$ the potential temperature Θ at the surface suddenly changes to Θ_1. Thus, we have in addition to the initial condition (84.1) the boundary condition that

$$\Theta = \Theta_1 \qquad \text{when} \qquad z = 0 \tag{84.2}$$

Finally, it will be assumed that the surface of the earth or of the ocean is not cooled by its contact with the cold air. This

[1] BRUNT, D., "Physical and Dynamical Meteorology," 2d ed., p. 228, Cambridge University Press, London, 1939.

condition is probably less well satisfied over land than over the ocean, where the cooled surface water will be replaced by warmer water from below owing to turbulent mixing. The potential temperature Θ must satisfy the differential equation for the turbulent mass exchange,

$$\frac{\partial \Theta}{\partial t} = \frac{A}{\rho} \frac{\partial^2 \Theta}{\partial z^2} \tag{84.3}$$

and the two conditions (84.1) and (84.2). In (84.3), A is again regarded as constant. In the theory of the conduction of heat, it is shown that the solution of (84.3) that satisfies (84.2) and (84.1) is given by

$$\Theta = \Theta_0 + \gamma z + (\Theta_1 - \Theta_0) \left\{1 - E\left[\frac{z}{\sqrt{4(A/\rho)t}}\right]\right\} \tag{84.4}$$

Here E stands for the *error function*,

$$E(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-x^2}\, dx \tag{84.5}$$

Tables of this function are available,[1] thus (84.4) can readily be evaluated for concrete examples.

The reader who is not familiar with the solution (84.4) can easily verify that it satisfies the initial condition (84.1) for $t = 0$, the boundary condition (84.2) for $z = 0$, and the differential equation (84.3).

For the following discussion, it is useful to keep in mind that the increase in potential temperature $\Delta\Theta$ and in actual temperature ΔT due to turbulent transfer of heat is related by the formula

$$\Delta T = \Delta\Theta \left(\frac{p}{P}\right)^{\kappa}$$

according to (9.1), provided that the pressure p at the level under consideration does not change. Under average conditions the following table applies:

[1] For instance, F. Linke, "Meteorologisches Taschenbuch," 2d ed., Table 93, Akademische Verlagsgesellschaft, Leipzig, 1933; Peirce, B. O., "A Short Table of Integrals," 3d ed., pp. 116–120, Ginn and Company, Boston, 1929.

Height, km	0	1	2	3
$(p/P)^\kappa$	1	0.97	0.94	0.91

Therefore, the variation of the actual temperature is very closely represented by the variation of the potential temperature.

At the surface, the total effect of the heating that is given by the difference $\Theta_1 - \Theta_0$ becomes noticeable immediately, at greater heights theoretically only after an infinite time has elapsed. To obtain a first orientation about the propagation of the heating upward, we may determine the height z_m at which the potential temperature, after a given time t, has increased by $(\Theta_1 - \Theta_0)/2$, *i.e.*, by one-half the total possible increase. By analogy with the "mean life" of radioactive substances, this height z_m may be called *the height of mean heating.* The time after which Θ has risen by $(\Theta_1 - \Theta_0)/2$, at a given level, may be called *the time of mean heating.* Upon formulating the condition for the height of mean heating with the aid of (84.4), it follows that

$$\Theta - (\Theta_0 + \gamma z_m) = \frac{\Theta_1 - \Theta_0}{2} = (\Theta_1 - \Theta_0)\left\{1 - E\left[\frac{z_m}{\sqrt{4(A/\rho)t}}\right]\right\}$$

or

$$E\left[\frac{z_m}{\sqrt{4(A/\rho)t}}\right] = \frac{1}{2}$$

According to tables of the error function E,

$$z_m = 0.477\sqrt{4\frac{A}{\rho}t} \tag{84.6}$$

and therefore the height of mean heating can be found for any time when A is given.

In Fig. 63 the increase of the height of mean heating with time is shown for different values of A. It has been assumed that $\rho = 10^{-3}$ gm/cm^3 as a mean value. The heating effect spreads rapidly through the lowest layers, but it proceeds more slowly at higher altitudes. When $A = 50$ gm cm^{-1} sec^{-1}, for instance, the height of mean heating is 1000 m after 2½ days, but only after 10 days has it reached 2000 m. When observations of the temperature increase with time at different heights are available, Fig. 63 may be used to estimate the value of A.

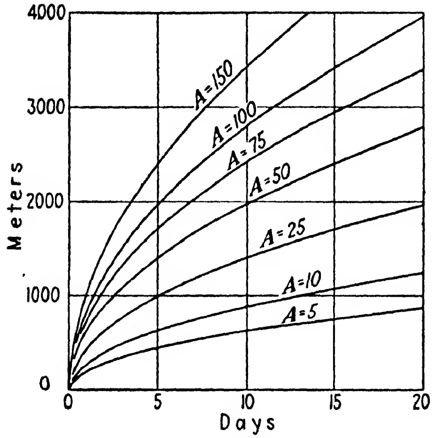

FIG. 63.—Height of mean heating for various coefficients of turbulent mass exchange (A in cgs units).

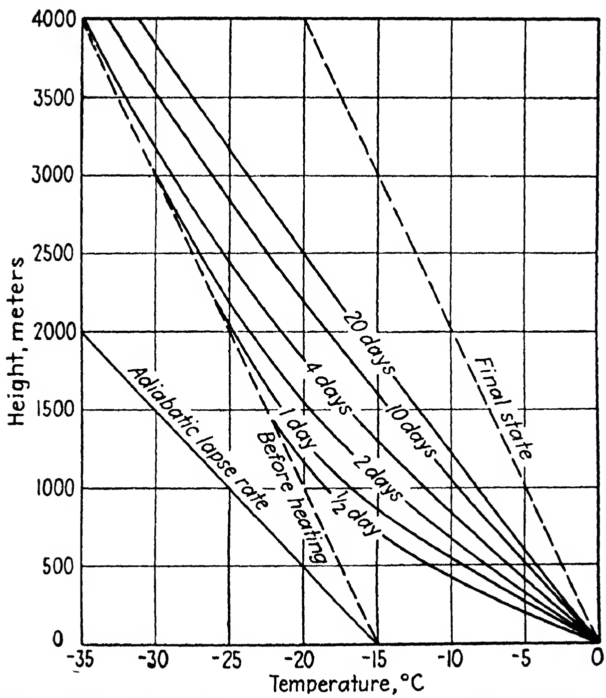

FIG. 64.—Heating of a cold air mass from below. (*After Schwerdtfeger.*)

Schwerdtfeger[1] has given an example (Fig. 64) of the vertical distribution of Θ when the air is heated from the ground. It is assumed that $\partial\Theta/\partial z = 0.5°\text{C}/100$ m, $\Theta_0 = -15°\text{C}$, $\Theta_1 = 0°\text{C}$, $A = 40$ gm cm^{-1} sec^{-1}, and $\rho = 10^{-3}$ gm/cm^3. After the heating has begun, the potential temperature decreases from the surface upward to a minimum. At higher levels, Θ increases in the same manner as before the heating began. This minimum moves higher and becomes less sharp as time goes on.

The lowest layer of the air mass under consideration attains a superadiabatic lapse rate during the first days of heating. Its equilibrium is thus unstable, whereas the layers above the level of minimum potential temperature are stable. Obviously, the assumption that A = const throughout the whole air mass is therefore untenable. Nevertheless, the preceding calculations give at least a schematic picture of the role that turbulent mass exchange may play in the transformation of air masses. Schwerdtfeger's results are represented in Fig. 64; but instead of the potential temperature the temperature has been plotted as a function of the height.

The problem of a warm air mass moving across a cold surface has been treated by Taylor.[2] The solution is the same as (84.4) except that $\Theta_1 - \Theta_0$ is now negative. Owing to the cooling of the lowest layers of the atmosphere by turbulent mass exchange, an inversion is formed. This inversion attains higher altitudes as the cooling spreads upward. Taylor applies his results to temperature observations above the Newfoundland Banks. Here, the air that comes from the warmer land is cooled in contact with the cold oceanic water, and an inversion is formed. From the height of this inversion and from the estimated time during which the air must have been flowing over the water Taylor obtained values for A of the order 10^0 gm cm^{-1} sec^{-1}. This small value is caused by the very stable stratification of the atmosphere.

85. Lateral Mixing and Its Study by Isentropic Analysis. Equation (81.4) shows that the transport of a property in a given direction is proportional to its gradient in this direction and

[1] SCHWERDTFEGER, W., *Veröffentlich. Geophys. Inst. Leipzig,* 2d ser., **4,** 253, 1931. More examples will be found in W. Schmidt, "Der Massenaustausch in freier Luft und verwandte Erscheinungen."

[2] TAYLOR, G. I., *Phil. Trans. Roy. Soc. A,* **215,** 1, 1915.

to the coefficient of turbulent mass exchange. If the latter is the same in all directions, the transport must be almost exclusively in the vertical direction, as was assumed in Sec. 81, for all meteorological elements change much more quickly in this direction than in the horizontal.[1]

The coefficient of turbulent mass exchange in the vertical direction is roughly of the order of magnitude 10^1 to 10^2 gm cm^{-1} sec^{-2}. Rossby has suggested that the coefficient of turbulent mass exchange in the horizontal direction may be very much larger than that of vertical mass exchange and that therefore eddy viscosity and eddy conductivity might have a great effect, also, on the horizontal distribution of the properties of the air, especially on the momentum of motion. For the ocean the importance of this "lateral mixing" appears well established.[2]

Rossby[2] has pointed out that owing to the strong gravitational stability of the atmosphere shown by the increase of the potential temperature with the altitude the eddy motions responsible for lateral mixing will take place along surfaces of equal potential temperature rather than along horizontal surfaces, which as a rule intersect the surfaces of equal potential temperature. Because the surfaces of equal potential temperature are also surfaces of constant entropy according to Eq. (22.5), at least for unsaturated air, they may also be called "isentropic" surfaces.

In order to study the eddy motions responsible for lateral mixing and the flow patterns in the free air in general, Rossby and his collaborators[3] have developed a new method, the so-called "isentropic analysis," following a suggestion originally made by Shaw.[4] Isentropic analysis assumes that the air currents in the free atmosphere move with and within their proper isentropic surfaces rather than in horizontal planes, an assumption that may be justified at least over reasonably short periods of time and as long as nonadiabatic effects such as radiation, condensa-

[1] Hesselberg, T., and Friedmann, A., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **1**, 147, 1914.

[2] Rossby, C.-G., and collaborators, *Bull. Am. Met. Soc.*, **18**, 201, 1937.

[3] See, especially, J. Namias, in S. Petterssen, "Weather Analysis and Forecasting," Chap. VIII, McGraw-Hill Book Company, Inc., New York, 1940. Or, J. Namias, Air Mass and Isentropic Analysis, X, *Am. Met. Soc.*, Milton, Mass., 1940.

[4] Shaw, N., Manual of Meteorology, vol. 3, 259, Cambridge University Press, London, 1933.

tion, and evaporation can be neglected. Apart from the aerological observations of pressure, temperature, and humidity the upper winds as determined from pilot-balloon observations can be of assistance in the construction of isentropic charts, as pointed out by Spilhaus.[1] To determine the intensity of lateral mixing and to find the trajectories of the air from day to day the distribution of the specific humidity is plotted on the isentropic charts. We have thus two "coordinates," potential temperature and specific humidity, to follow the motion of a parcel of air. For a complete determination of the position a third conservative property is required. Starr and Neiburger[2] have shown that the so-called "potential vorticity" may be used for this purpose.

This quantity has been introduced by Rossby.[3] According to Bjerknes's circulation theorem (52.2),

$$C + 2\omega F = \text{const}$$

in an autobarotropic fluid. Therefore, the absolute vorticity, *i.e.*, the vorticity relative to the earth plus the vorticity due to the earth's rotation, remains constant. In general the stratification of the atmosphere is not autobarotropic, but it is possible to subdivide the atmosphere into sufficiently small layers each of which may be regarded as autobarotropic. If the vertical variations of u and v are neglected and if it is assumed that the hydrostatic equation (6.1) is satisfied, it follows from (47.2) that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\omega \sin\varphi\, v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{85.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\omega \sin\varphi\, u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{85.11}$$

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{6.1}$$

Owing to the barotropic relation $\rho = \rho(p)$,

$$\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{d\rho}{dp}\frac{\partial p}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{d\rho}{dp}\frac{\partial p}{\partial x}$$

Upon differentiating (85.11) partially with respect to x, and (85.1) with respect to y, it is found that

[1] Spilhaus, A., *Bull. Am. Met. Soc.*, **21**, 239, 1940.

[2] Starr, V. P., and Neiburger, M., *J. Mar. Research*, **3**, 202, 1940.

[3] Rossby, C.-G., *Quart. J. Roy. Met. Soc.*, **66**, Suppl. 68, 1940.

$$\frac{d\zeta}{dt} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) 2\omega \sin \varphi + (2\omega \sin \varphi + \zeta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

where

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{d}{dt}, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta$$

Noting that

$$\frac{d}{dt}(2\omega \sin \varphi) = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) 2\omega \sin \varphi$$

the preceding equation is reduced to

$$\frac{d}{dt}(\zeta + 2\omega \sin \varphi) + (\zeta + 2\omega \sin \varphi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (85.2)$$

The equation of continuity (47.3) may, in view of the present meaning of d/dt, be written in the form

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \rho w}{\partial z} = 0$$

Multiplying by $g\,dz$ and integrating from the lower boundary z_0 to the upper boundary z_1 of the autobarotropic layer,

$$\int_{z_0}^{z_1} g\,dz\frac{d\rho}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_{z_0}^{z_1} \rho g\,dz + \int_{z_0}^{z_1} g\frac{\partial \rho w}{\partial z}\,dz = 0$$

It must be noted that z_0 and z_1 may here vary with the time and with x and y, contrary to page 159 where a fixed level was considered. Thus

$$\int_{z_0}^{z_1} g\frac{\partial \rho w}{\partial z}\,dz = g[(\rho w)_1 - (\rho w)_2] = g\left(\rho_1\frac{dz_1}{dt} - \rho_2\frac{dz_2}{dt}\right)$$

Further,

$$\frac{d}{dt}\int_{z_0}^{z_1} g\rho\,dz = \int_{z_0}^{z_1} g\frac{d\rho}{dt}\,dz + g\left(\rho_1\frac{dz_1}{dt} - \rho_2\frac{dz_2}{dt}\right)$$

and

$$\int_{z_0}^{z_1} g\rho\,dz = p_1 - p_0$$

Thus, the equation of continuity changes into

$$\frac{d}{dt}(p_0 - p_1) + (p_0 - p_1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Upon substituting the expression for $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ in (85.2), it follows that

$$\frac{\zeta + 2\omega \sin \varphi}{p_0 - p_1} = \text{const} \tag{85.3}$$

Although the relative vorticity ζ changes with the latitude and with the mass $p_0 - p_1$ of the layer, the expression (85.3) remains constant. Thus, if the air is brought to a standard latitude φ_S and if the pressure difference $p_0 - p_1$ is reduced to a standard Δp_S, the resulting vorticity ζ_S may be considered as a characteristic, conservative property, the potential vorticity. The actual determination of the vorticity is, of course, difficult in view of the unreliability of the wind observations, but some progress has been made by Starr and Neiburger.

Grimminger[1] has studied the gradual spread of the lines of equal specific humidity sideways in moist currents as they appear on isentropic charts. Under the assumption that this spread is due to lateral mixing, he finds coefficients of lateral mixing between 4×10^5 and 5×10^7 gm cm^{-1} sec^{-2}. These figures are at least 10^3 times larger than the coefficients of eddy viscosity in the vertical. Because the vertical gradients of the temperature and of the horizontal-wind velocity, to mention only two elements, are only about 10^2 times larger than the horizontal gradients, the importance of the lateral mixing as compared with vertical mixing would be clearly established if the order of magnitude of Grimminger's figures could be regarded as final.

In view of the suggested importance of lateral mixing, Rossby[2] has given particular attention to the dynamics of the jet stream and its applications in meteorology and oceanography. To

[1] Grimminger, G., *Trans. Am. Geophys. Un., 19th Ann. Meeting*, p. 163, 1938.

[2] Rossby, C.-G., Papers in Physical Oceanography and Meteorology, Mass. Inst. Tech. and Woods Hole Ocean. Inst., **5**, 1, 1936; *J. Mar. Research*, **1**, 15, 1937.

obtain an idea of the dynamics of such a current without entering into the details of the mathematical discussion, consider a current through a fluid originally at rest, as shown in Fig. 65. The direction downstream is indicated by the full arrow in the center of the figure. As long as the effect of lateral mixing at the boundaries of this current is disregarded, the Coriolis force of this motion must be balanced by a pressure gradient at right angles to the current, with the higher pressure to the right. Outside the current the pressure is uniform.

Owing to lateral mixing the fluid masses to the right and to the left of the original current must receive a slight acceleration downstream while the original current loses speed, as indicated by the broken arrows in Fig. 65. Thus, velocities in excess of the

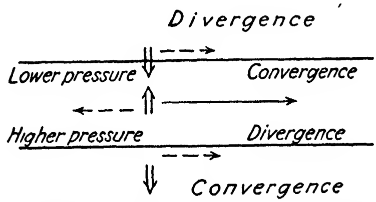

Fig. 65.—Effect of lateral mixing on a current through a fluid at rest.

geostrophic values are established to the right and left of the current axis, whereas the velocity in the axis is less. The Coriolis force of these excess velocities is not balanced by the original pressure gradient. Consequently, transversal velocity components are produced as indicated by the double-shaft arrows in Fig. 65. Therefore, in the left half of the current system, convergence and pressure rise must occur; in the right half, divergence and pressure fall. The mass distribution becomes gradually more adjusted to the new velocity distribution produced by lateral mixing. Farther away from the axis the transversal velocities must vanish. Thus, a region of divergence is found at greater distances from the axis on the left side, a region of convergence on the right side. A trough of low pressure will form to the left, a ridge of high pressure to the right. Rossby considers quantitatively the development of this current and its pressure field, also. It appears that the current system is not able, by means of the mechanism considered, to build up such compensating pressure gradients as could offset completely the Coriolis forces of the motion produced by lateral mixing. As a

result the current systems of the atmosphere, and similarly of the oceans, have the tendency to break up into large-scale anticyclonic eddies to the right of the current and into cyclonic eddies to the left.

Problems

26. Determine the vertical distribution of dust particles in the steady case, assuming the same constant sinking velocity for all dust particles and regarding the density of the air as independent of the altitude.

a. When the coefficient of turbulent mass exchange is constant.

b. When the coefficient of turbulent mass exchange has a finite value at the ground and increases linearly with the altitude.

27. An air mass in contact with a cold surface is cooled from below by turbulent mass exchange. Assume that the surface temperature and the coefficient of turbulent mass exchange are constant and that the temperature decreases linearly with the altitude before the cooling. At which height z_{max} has the temperature a maximum at a given time? For how long after the beginning of the cooling does an inversion exist? At which time is the altitude z_{max} of the maximum temperature highest? Determine the expression for z_{max} and the temperature at z_{max} at this time [Note: The approximate relation (9.31) $\Theta = T + \Gamma z$ may be used for the relation between the potential temperature and the temperature.]

28. Find an expression for the daily temperature period at different altitudes when the coefficient of turbulent mass exchange is independent of the altitude but is a periodic function of the time having its maximum at the time of maximum temperature and its minimum 12 hr. later.

CHAPTER XII

THE ENERGY OF ATMOSPHERIC MOTIONS

86. The Amount of Available Energy. The energy required to set the atmosphere in motion and to maintain the air currents in spite of the effect of viscosity is ultimately of solar origin. The manner in which the solar energy is received at the surface of the earth and in the atmosphere has been described in Chap. V. According to page 94 the amount of solar energy received on the average over the whole earth is 0.276 cal/cm^2 min if Aldrich's value of the albedo is adopted. This is in mechanical units 0.0193 watt/cm^2.[1]

87. The Atmospheric-energy Equation. Upon multiplying the three equations of motion (47.2), respectively, by u, v, and w and adding, it follows that

$$\frac{d}{dt}\frac{u^2+v^2+w^2}{2} = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) + (uF_x + vF_y + wF_z)$$

The components of the external force F_x, F_y, F_z may now represent the components of the frictional force. Furthermore, the vertical component F_z contains, of course, the acceleration of gravity $-g$ which will be written down separately. Because $w = dz/dt$, the preceding equation may be written

$$\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2} + gz\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) + (uF_x + vF_y + wF_z) \quad (87.1)$$

$\dfrac{u^2+v^2+w^2}{2}$ is the kinetic energy, and gz the potential energy, both per unit of mass. If s is the direction of the velocity and c its intensity, it follows that

[1] The reader will remember that the unit of energy in the cgs system is the erg, 1 erg = 10^{-7} joule, and the unit of power, *i.e.*, the work done per unit time, is ergs/sec, 1 erg/sec = 10^{-7} watt.

$$u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = c\left[\cos(s,x)\frac{\partial p}{\partial x} + \cos(s,y)\frac{\partial p}{\partial y} + \cos(s,z)\frac{\partial p}{\partial z}\right] = c\frac{\partial p}{\partial s} \quad (87.11)$$

where $\partial p/\partial s$ is the component of the pressure gradient in the direction of the velocity. Similarly, for the frictional force,

$$uF_x + vF_y + wF_z = cF_s$$

Thus, the two expressions on the right-hand side of (87.1) represent the work done per unit time by the pressure forces and frictional forces.

To obtain an energy equation that comprises also the variations of the internal energy, (87.1) may be combined with the first law of thermodynamics (8.1) for an ideal gas,

$$J\frac{dq}{dt} = Jc_v\frac{dT}{dt} + p\frac{d(1/\rho)}{dt}$$

Equation (8.1) is here multiplied by the mechanical equivalent of heat in order to express all forms of energy in mechanical units. Upon adding (8.1) in its above form to (87.1), it follows that

$$J\frac{dq}{dt} = \frac{d}{dt}\left(\frac{c^2}{2} + gz + Jc_vT\right) + p\frac{d(1/\rho)}{dt} + \frac{1}{\rho}c\frac{\partial p}{\partial s} - cF_s \quad (87.2)$$

This equation relates the variation of the heat content of an (ideal) gas and the changes of its kinetic, potential, and internal energy. It shows further that the energy spent by expansion, by motion across the isobars, and the energy loss due to friction have to be considered, also. It should be noted that (87.2) refers to an individual fluid particle. Because

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

it follows that

$$c\frac{\partial p}{\partial s} = \frac{dp}{dt} - \frac{\partial p}{\partial t}$$

and Eq. (87.2) can be written in the following form:

$$J\frac{dq}{dt} = \frac{d}{dt}\left(\frac{c^2}{2} + gz + Jc_vT + \frac{p}{\rho}\right) - \frac{1}{\rho}\frac{\partial p}{\partial t} - cF_s \qquad (87.3)$$

If the friction may be neglected, $F_s = 0$. When the motion is steady, $\partial p/\partial t = 0$; and when the changes of state are adiabatic, $dq/dt = 0$. With these three assumptions, (87.3) may be integrated with respect to the time,[1]

$$\frac{c^2}{2} + gz + \frac{p}{\rho} + Jc_vT = \text{const} \qquad (87.4)$$

This equation holds for successive states of an individual particle or for simultaneous states of different particles on the same streamline; for the motion was assumed to be steady, and thus the streamlines and the paths of the particles coincide (page 134). But the constant in (87.4) may, of course, vary from streamline to streamline.

Equation (87.4) is a generalization of "Bernoulli's equation"

$$\frac{c^2}{2} + gz + \frac{p}{\rho} = \text{const} \qquad (87.41)$$

for an incompressible fluid. It shows, for instance, that when the velocity of the particle changes, the pressure must change in the opposite direction. This explains why an obstacle in the air flow, such as a building, may produce a noticeable reduction of the barometric pressure in the case of high wind velocities. Koschmieder[2] has given examples of measurements in the observatory on the mountain Schneekoppe where the pressure is reduced 1 mb when the wind velocity is 17 m/sec and 2 mb when it is 24 m/sec. The effects vary with the different conditions under which the barometer is placed, and a separate investigation is necessary for each observatory.

Topographic obstacles such as mountains give rise to a similar pressure effect. Koschmieder[3] has shown how the topographic disturbances of the pressure field can be computed from (87.4) provided that the velocity distribution is known. Examples of

[1] Bjerknes, V., *Met. Z.*, **34**, 166, 1917.
[2] Koschmieder, H., *Met. Z.*, **47**, 317, 1930.
[3] Koschmieder, H., *Met. Z.*, **43**, 246, 1926.

such obstacles consisting of promontories projecting out to sea and the resulting disturbances have been described by Crossley.[1]

88. The Energy of Air Columns. The internal energy of an air column of unit cross section extending to the level h,

$$I^* = Jc_v \int_0^h T\rho \, dz \tag{88.1}$$

where I^* is expressed in mechanical units. With the aid of the hydrostatic equation (6.1),

$$I^* = \frac{Jc_v}{g} \int_{p_h}^{p_0} T \, dp \tag{88.2}$$

This formula may sometimes be more convenient for the computation of the internal energy.

The potential energy of the same air column,

$$P^* = \int_0^h gz\rho \, dz \tag{88.3}$$

A simple relation exists between P^* and I^*. By the hydrostatic equation (6.1) and by integration by parts,

$$P^* = \int_{p_h}^{p_0} z \, dp = -p_h h + \int_0^h p \, dz$$

Because $p = R\rho T$,

$$P^* = -p_h h + R \int_0^h T\rho \, dz$$

Thus

$$P^* = -p_h h + \frac{R}{Jc_v} I^* \tag{88.4}$$

or because $\dfrac{R}{Jc_v} = \dfrac{c_p - c_v}{c_v} = \lambda - 1$, according to Sec. 8,

$$P^* = -p_h h + (\lambda - 1) I^* \tag{88.5}$$

It should be noted that in these relations the latent heat of condensation is not considered.

89. The Dissipation of Energy. The rate of dissipation of the kinetic energy of atmospheric motion is represented by the term $-cF_s$ in (87.2). To obtain an estimate of this term, it may

[1] Crossley, A. F., *Quart. J. Roy. Met. Soc.*, **64**, 477, 1938.

be assumed that the vertical component w of the velocity can be neglected and that

$$F_x = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \qquad \text{and} \qquad F_y = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)$$

as follows from (71.5) and Sec. 74. Upon multiplying by ρ and integrating from the surface to infinity, it follows that the rate of dissipation in an air column of unit cross section,

$$\Delta = \int_0^\infty \left[u\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + v\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)\right] dz \tag{89.1}$$

Upon integrating by parts and assuming that at infinity the velocity vanishes, it follows that

$$\Delta = -\mu\left(u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z}\right)_{z=0} - \int_0^\infty \mu\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] dz \tag{89.2}$$

The first term on the right-hand side represents the effect of surface friction, the second the dissipation of energy within the air column.[1] The latter term appears in the denominator of the Richardson number (page 200). Therefore, Δ represents the total effect of friction at the surface and in the free atmosphere.

To simplify matters, it will be assumed that μ and the pressure gradient are constant and that the velocity does not change with time. This implies, of course, that the losses due to the dissipation of energy are continuously replaced. It follows then, from (89.1) and (76.1), that

$$\Delta = +\int_0^\infty \frac{\partial p}{\partial x} u\, dz = +2\omega \sin\varphi\, v_g \int_0^\infty \rho u\, dz$$

where v_g is the geostrophic wind velocity. Above the gradient-wind level D, u is very small so that the dissipation occurs mainly in the layer from the surface to D. Therefore, only the dissipation Δ_D in the air column up to D will be computed. The density ρ may then be considered as constant without causing a serious error. Because only a rough estimate of Δ is desired, it will further be assumed that the wind velocity vanishes at the

[1] Hesselberg, T., *Geofys. Pub.*, **3**, 5, 1924.

earth's surface. Then, $\alpha_0 = 45°$ according to page 204; and, from (76.7),

$$u = -v_g e^{-az} \sin az$$
$$v = v_g(1 - e^{-az} \cos az)$$

The gradient-wind level $D = \pi/a$ according to (76.71). When the surface-wind velocity vanishes, it follows from (89.2) that the dissipation due to surface friction is zero, but the second term in (89.2) becomes larger because the increase of the wind velocity with altitude is greater. The value found for Δ when $u = v = 0$ at the surface is therefore a reasonable estimate of the total dissipation. Upon substituting the value for u, it follows that

$$\Delta_D = 2\omega \sin \varphi\, v_g \rho \int_0^D u\, dz = -\frac{\omega \sin \varphi\, \rho}{\pi} D v_g^2 (1 + e^{-\pi}) \quad (89.3)$$

If the integration had been extended to infinity, the term $e^{-\pi}$ (= 0.0432) would have dropped out. The dissipation Δ_D may be compared with the kinetic energy K_D^* contained in the same volume,

$$K_D^* = \frac{\rho}{2} \int_0^D (u^2 + v^2)\, dz = \frac{\rho}{2} v_g^2 D \left[1 - \frac{(1 + e^{-\pi})^2}{2\pi}\right]$$
$$= \frac{\rho}{2} v_g^2 D (1 - 0.173) \quad (89.4)$$

Thus, the ratio

$$\frac{\Delta_D}{K_D^*} = -\frac{2\omega \sin \varphi}{\pi} \frac{1.043}{0.827} = -2\omega \sin \varphi\, 0.401. \quad (89.5)$$

At a latitude of about 43.4°, for instance, the ratio would be 0.4×10^{-4} per sec, or 0.144 per hr. This figure agrees well with more accurate estimates by Sverdrup.[1] In approximately 7 hr the kinetic energy of the frictional layer would be dissipated if it were not replaced from other sources. If the gradient level is assumed to be at 1000 m, the frictional layer comprises about one-tenth the total mass of the atmosphere and, the usual increase of the wind velocity being disregarded, also about one-tenth of the total kinetic energy. When the kinetic energy of the upper layers is used to replace the kinetic energy in the

[1] Sverdrup, H. U., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **2**, 190, 1918.

frictional layer, it would take about 3 days before the energy is dissipated provided that the rate of dissipation remains the same.

If $v_g = 10$ m/sec, $D = 1000$ m, $\rho = 1.2 \times 10^{-3}$ gm/cm³, $K^* = 5$ joules/cm², according to (89.4). Thus, $\Delta_D = 2 \times 10^{-4}$ watt/cm². This is about 1 per cent of 0.0193 watt/cm², the average amount of solar radiation received at the top of the atmosphere (Sec. 86) after deducting the losses due to the albedo. Sverdrup[1] found by a much more thorough discussion that about 2 per cent of the solar radiation is used to replace the losses due to friction; thus, the figure derived here by a very simple consideration gives at least the right order of magnitude.

90. The Energy Transformations in a Closed System. When two air masses of different temperatures are lying side by side or when the potentially colder air lies over the potentially warmer air, the stratification of the air represents a certain amount of potential and internal energy. Margules[2] has shown that this energy may be of the right order of magnitude to account for the observed kinetic energy. In order to make possible a mathematical discussion of the problem, he assumes that the two air masses form a "closed system." This implies that the two air masses are enclosed by fictitious walls through which heat or energy in any other form cannot pass and that the velocity component normal to these walls vanishes.

In deriving the energy equation for such a closed system, it is necessary to show that

$$\int \frac{dL}{dt} \rho \, d\tau = \frac{\partial}{\partial t} \int L\rho \, d\tau \tag{90.1}$$

Here L is a mass property of the fluid such as kinetic energy, $d\tau$ is a volume element, and the integrations are extended over the whole closed system. Because d/dt stands for the individual variation, it follows that

$$\int \frac{dL}{dt} \rho \, d\tau = \int \frac{\partial L}{\partial t} \rho \, d\tau + \int \left(u \frac{\partial L}{\partial x} + v \frac{\partial L}{\partial y} + w \frac{\partial L}{\partial z} \right) \rho \, d\tau$$

or because the order of the operations $\partial/\partial t$ and $\int$ may be interchanged,

[1] Sverdrup, *loc. cit.*

[2] Margules, M., *Jahrb. Zent.-Anst. Met. Geodynamik*, **40**, Suppl. 2, 1903, Vienna, 1905; *Met. Z.*, **23**, 481, 1906.

$$= \frac{\partial}{\partial t} \int L\rho \, d\tau - \int L \frac{\partial \rho}{\partial t} \, d\tau + \int \left(u \frac{\partial L}{\partial x} + v \frac{\partial L}{\partial y} + w \frac{\partial L}{\partial z} \right) \rho \, d\tau$$

According to the equation of continuity (47.3),

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right)$$

Thus

$$\int \frac{dL}{dt} \rho \, d\tau = \frac{\partial}{\partial t} \int L\rho \, d\tau + \int \left[\frac{\partial (L\rho u)}{\partial x} + \frac{\partial (L\rho v)}{\partial y} + \frac{\partial (L\rho w)}{\partial z} \right] d\tau$$

The second integral may be transformed by means of Green's theorem

$$\int \left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \right) d\tau = \int U_n \, d\sigma \qquad (90.2)$$

U_n is the component of U perpendicular to the boundary surface of the volume under consideration; $d\sigma$ is an element of this surface. U_x, U_y, U_z are the components of U parallel to the x-, y-, and z-axes. The reader who is not familiar with the theorem (90.2) will have no difficulty in seeing its validity if he considers U as the velocity multiplied by the density. The equation states, then, that the divergence (see page 135) of the mass in the closed system equals the mass transport through its boundary. Applying (90.2) to the preceding equation, it follows that

$$\int \left(\frac{\partial L\rho u}{\partial x} + \frac{\partial L\rho v}{\partial y} + \frac{\partial L\rho w}{\partial z} \right) d\tau = \int L\rho c_n \, d\sigma$$

where c_n is the velocity component normal to the boundary surface. c_n vanishes according to the definition of the closed system. Equation (90.1) is thus proved.

To derive Margules's energy equation for a closed system, (87.2) may be multiplied by $\rho \, d\tau$ and integrated over the volume of the system. Then, because $\int \rho \frac{dq}{dt} \, d\tau = 0$ for a closed system, with the aid of (90.1),

$$\frac{\partial}{\partial t} (K^* + P^* + I^*) + \int \left[\rho p \frac{d(1/\rho)}{dt} + c \frac{\partial p}{\partial s} \right] d\tau - \int \rho c F_s d\tau = 0$$

Here

$$K^* = \int \rho \frac{c^2}{2}\, d\tau, \text{ the kinetic energy}$$

$$P^* = \int g\rho z\, d\tau, \text{ the potential energy}$$

$$I^* = \int Jc_v\rho T\, d\tau, \text{ the internal energy}$$

Further,

$$\rho p \frac{d(1/\rho)}{dt} + c\frac{\partial p}{\partial s} = -\frac{p}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

or, with the equation of continuity (47.3),

$$= +\frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} + \frac{\partial(pw)}{\partial z}$$

Therefore,

$$\int \left(\rho p\frac{d(1/\rho)}{dt} + c\frac{\partial p}{\partial s}\right) d\tau = +\int pc_n\, d\sigma$$

according to (90.2). The last integral vanishes in a closed system. Thus, the energy equation for closed systems becomes, when the effects of friction are disregarded,

$$K^* + P^* + I^* = \text{const} \tag{90.3}$$

If (88.5) is taken into account, this equation may be written in the form

$$K^* - hp_h + \lambda I^* = \text{const} \tag{90.31}$$

Consider a closed system in which the air is at rest. Let its potential plus internal energy be $P^* + I^*$. If a rearrangement of the stratification occurs, the sum of potential and internal energy becomes $P^{*\prime} + I^{*\prime}$. If the rearrangement occurs spontaneously, the sum of potential and internal energy must decrease so that a certain amount of kinetic energy $K^{*\prime}$ may be produced. If the small variations of the height h are neglected, it follows that

$$K^{*\prime} = \lambda(I^* - I^{*\prime}) \tag{90.4}$$

If M is the mass of the system and $\bar{c}$ the mean velocity that it acquires by rearrangement,

$$K^{*\prime} = \frac{M}{2}\bar{c}^2 \tag{90.5}$$

so that $\bar{c}$ can be computed. The value of $\bar{c}$ obtained in this manner is a maximum value, for some of the kinetic energy generated will be used up by friction. An example of such an energy computation will be given in Sec. 91 to show how Margules has solved such problems.

91. The Energy of Air Masses of Different Temperature Lying Side by Side. We shall consider[1] two air masses of different temperature lying side by side (Fig. 66). The surface pressure and temperature in the colder mass 1 are p_{01} and T_{01}, and the pressure and temperature at the top of the layer p_h and T_{h1}. The same quantities for the warmer mass are denoted by the index 2, except the pressure at the top of the layer which is the same for both masses. To simplify matters, it is assumed that the lapse rate of temperature is adiabatic in both layers and that the changes occur adiabatically. The horizontal cross sections of both masses may be the same, say $B/2$.

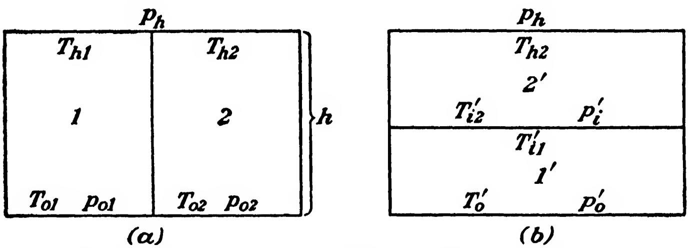

FIG. 66.—Rearrangement of two air masses of different temperature. (*After Margules.*)

The sum of potential and internal energy of the system attains a minimum, and therefore the kinetic energy reaches a maximum, when the colder mass comes to be situated under the warmer mass, as indicated in Fig. 66*b*. The cross section of each mass is now B. It may be assumed that the strata in each layer retain their position relative to each other during the transformation. Because the mass of each layer remains constant but spreads over

[1] MARGULES, *loc. cit.*

twice its former cross section the pressure p_2' at any level in the second mass after rearrangement is given by the relation

$$p_2' - p_h = \tfrac{1}{2}(p_2 - p_h)$$

Similarly, in the first mass,

$$p_1' - p_h = \frac{p_1 - p_h}{2} + \frac{p_{02} - p_h}{2}$$

Thus,

$$p_2' = p_2 - \frac{p_2 - p_h}{2} \tag{91.1}$$

$$p_1' = p_1 + \frac{p_{02} - p_1}{2} \tag{91.11}$$

With the aid of these equations the pressures p_0' at the ground and p_{i1}' at the discontinuity can be found. The temperature T_1' and T_2' in both layers at any level can be expressed by the adiabatic equation (8.41),

$$T_1' = T_1 \left(\frac{p_1'}{p_1}\right)^{\kappa} \tag{91.2}$$

$$T_2' = T_2 \left(\frac{p_2'}{p_2}\right)^{\kappa} \tag{91.21}$$

for the relative order within the layers is supposed to be undisturbed during the rearrangement. The temperatures T_0', T_{i1}', and T_{i2}' at the ground and on both sides of the discontinuity can be found by means of Eqs. (91.2) and (91.21). By the formula for the internal energy derived in Prob. 29 and Eqs. (90.4) and (90.5) the kinetic energy K^* and the mean velocity can be found. The numerical calculation is very laborious; for $K^{*\prime}$ appears as the small difference of two large figures, and thus the computation has to be done very accurately.

Margules, however, has also developed a much simpler approximation formula which gives good results. This formula will be derived here.

As long as the layers are not too thick, it follows from (91.1), (91.11), (91.2), and (91.21) that, approximately,

$$T_1' = T_1 \left(1 + \kappa \frac{p_{02} - p_1}{2p_1}\right) \tag{91.22}$$

$$T_2' = T_2 \left(1 - \kappa \frac{p_2 - p_h}{2p_2}\right) \tag{91.23}$$

The internal energy of the system in the original position,

$$I^* = \frac{Jc_v}{2g} B \left(\int_{p_h}^{p_{01}} T_1 \, dp_1 + \int_{p_h}^{p_{02}} T_2 \, dp_2 \right)$$

and in the final position,

$$I^{*\prime} = \frac{Jc_v}{g} B \left(\int_{p_i'}^{p_0'} T_1' \, dp_1' + \int_{p_h}^{p_i'} T_2' \, dp_2' \right)$$

according to (88.2).

The two integrals in the expression for $I^{*\prime}$ may be transformed with the aid of (91.1), (91.11), (91.22), and (91.23)

$$\int_{p_i'}^{p_0'} T_1' \, dp_1' = \frac{1}{2} \int_{p_h}^{p_{01}} T_1 \left(1 + \kappa \frac{p_{02} - p_1}{2p_1} \right) dp_1$$

$$\int_{p_h}^{p_i'} T_2' \, dp_2' = \frac{1}{2} \int_{p_h}^{p_{02}} T_2 \left(1 - \kappa \frac{p_2 - p_h}{2p_2} \right) dp_2$$

Therefore

$$I^* - I^{*\prime} = \frac{Jc_v}{4g} \kappa B \left(- \int_{p_h}^{p_{01}} T_1 \frac{p_{02} - p_1}{p_1} \, dp_1 + \int_{p_h}^{p_{02}} T_2 \frac{p_2 - p_h}{p_2} \, dp_2 \right) \qquad (91.3)$$

Even though only the first terms in the expansions (91.22) and (91.23) for T_1' and T_2' have been used, the last formula is correct to the terms of the second order, as shown by Margules. To evaluate the integrals, it may be noted that

$$\int_{p_h}^{p_{01}} T_1 \frac{dp_1}{p_1} = \frac{g}{R} h = \int_{p_h}^{p_{02}} T_2 \frac{dp_2}{p_2}$$

The mean temperature in each layer before the rearrangement may be introduced by the definition that

$$\int_{p_h}^{p_{01}} T_1 \, dp_1 = \overline{T_1}(p_{01} - p_h) \qquad \text{and} \qquad \int_{p_h}^{p_{02}} T_2 \, dp_2 = \overline{T_2}(p_{02} - p_h)$$

Thus, the difference of the internal energy before and after the rearrangement,

$$I^* - I^{*\prime} = \frac{Jc_v}{4g}\kappa B\left[-(p_{02} + p_h)\frac{g}{R}h + \overline{T_1}(p_{01} - p_h) + \overline{T_2}(p_{02} - p_h)\right] \quad (91.31)$$

This formula may be simplified still further, for it follows from (6.1) that, approximately,

$$p_{01} = p_h e^{\frac{g}{R}\frac{h}{T_1^*}} = p_h\left[1 + \frac{g}{R}\frac{h}{T_1^*} + \frac{1}{2}\left(\frac{g}{R}\frac{h}{T_1^*}\right)^2\right]$$

$$p_{02} = p_h e^{\frac{g}{R}\frac{h}{T_2^*}} = p_h\left[1 + \frac{g}{R}\frac{h}{T_2^*} + \frac{1}{2}\left(\frac{g}{R}\frac{h}{T_2^*}\right)^2\right]$$

T_1^* and T_2^* are suitably defined "barometric" mean temperatures,

$$\frac{h}{T} = \int_0^h \frac{dz}{T}$$

which differ slightly from the mean temperatures $\overline{T_1}$ and $\overline{T_2}$. But, for the purpose of the present calculation, they may be regarded as identical. The second-order terms in the series for p_{01} and p_{02} have been included, for the expression (91.3) for $I^* - I^{*\prime}$ is accurate to the second order. Upon substituting these approximations for p_{01} and p_{02} in (91.31) and neglecting a term of the third order the difference of the internal energies

$$I^* - I^{*\prime} = \frac{Jc_v}{4g}\kappa B p_h\left(\frac{g}{R}h\right)^2\frac{\overline{T_2} - \overline{T_1}}{2\overline{T_2}\,\overline{T_1}} \quad (91.4)$$

The mass of the two layers,

$$M = \frac{B}{g}\left(\frac{p_{02} - p_h}{2} + \frac{p_{01} - p_h}{2}\right) = \frac{B}{2}\frac{h}{R}p_h\frac{\overline{T_1} + \overline{T_2}}{\overline{T_2}\,\overline{T_1}}$$

for the second-order terms may be neglected here. According to (90.4) and (90.5),

$$\frac{M}{2}\bar{c}^2 = \lambda(I^* - I^{*\prime})$$

Hence, because $\lambda Jc_v\kappa/R = 1$ (Sec. 8),

$$\bar{c}^2 = \frac{1}{2}gh\frac{\overline{T_2} - \overline{T_1}}{\overline{T_2} + \overline{T_1}} \quad (91.5)$$

Because the lapse rate of temperature is the same in both layers, $\overline{T_2} - \overline{T_1} = T_{02} - T_{01} = \Delta T$. Further, let $\overline{T_2} + \overline{T_1} = 2T$. Then

$$\bar{c} = \frac{1}{2}\sqrt{gh\,\frac{\Delta T}{T}} \tag{91.6}$$

This is the maximum velocity obtainable when the two layers are arranged so that their potential and internal energy becomes a minimum.

Margules gives the following examples:

h, m	T, deg abs.	ΔT, deg C	c, m/sec	Approximate, m/sec
3000	246	5	12 2	12.2
3000	248	10	17 3	17.3
6000	216	5	18.3	18.4
6000	218	10	25.8	26.4

The last column gives the values of $\bar{c}$ computed with the approximation formula (91.6). The preceding column gives the figures computed with the exact formula. The agreement between both values is very good even for an air column 6000 m high, for which the assumptions on which the approximations are based are no longer very well satisfied.

The resulting wind velocities are quite considerable even when it is taken into account that they represent maximum values, for friction was not considered.

Margules's formula has been used by Schröder[1] to discuss the energy transformation in a regenerating cyclone. The cyclone studied by Schröder was already occluded when, through the infusion of fresh polar air, it acquired new potential and internal energy. With the lifting of the older polar air masses over the fresh polar air the newly acquired potential and internal energy decreases. Schröder shows that this new energy is sufficient to explain the observed increase of the kinetic energy of the cyclone even when allowance is made for the loss of energy through friction. It appears therefore that the kinetic energy of the motion in cyclones is gained by the transformation of potential and internal energy of air masses which originally were lying side

[1] SCHRÖDER, R., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **4**, 49, 1929.

by side. But Margules has already pointed out that his calculations do not show in which manner this transformation takes place, when the two air masses assume a more stable position. Starr[1] has extended Margules's theory in this respect under certain simplifying assumptions. He considers two incompressible fluid layers of different density lying side by side, as shown in Fig. 66*a*. In the final state of equilibrium the heavier mass must be situated under the warmer mass in the form of a wedge (page 171). The air must move at right angles to the vertical boundary of the juxtaposed layers in order to assume this position. This motion gives rise, also, to a velocity component parallel to the boundary. The final velocity distribution represents an amount of kinetic energy that is about one-third of the potential energy released during the readjustment. This according to Starr, is due to the assumption that the transition from the initial to the final state proceeds infinitely slowly, implying the existence of an external retarding agency which absorbs part of the available energy.

92. The Effect of Water Vapor on the Atmospheric-energy Transformations. The preceding discussions of the energy transformations in the atmosphere refer to dry air. Equations like (90.3) hold, of course, for moist air also, even when condensation takes place, for they state the principle of the conservation of energy; all that is necessary is that the same conditions—absence of friction, and adiabatic walls of the system under consideration—hold in the case of moist air as in the case of dry air. But the internal plus potential energy that is available for the production of kinetic energy may be very different in columns of dry air and of moist air, which have the same distribution of pressure and temperature. Thus, a single column of dry air is unstable provided that the temperature gradient is superadiabatic[2] but stable if it is less than adiabatic. On the other hand, if the air is moist, energy can be realized when the column of air overturns provided that it was conditionally unstable (page 57).

[1] Starr, V. P., *Monthly Weather Rev.*, **67**, 125, 1939.

[2] The computation of the energy to be realized when the layer assumes a stable stratification has been carried out by Littwin in H. Koschmieder, "Dynamische Meteorologie," p. 336, Akademische Verlagsgesellschaft, Leipzig, 1933, and by G. W. B. Normand, *Quart. J. Roy. Met. Soc.*, **64**, 71, 1938.

The important role of water vapor during transformations of atmospheric energy was stressed by Refsdal[1] and Raethjen.[2] Examples of the order of magnitude of the mean velocities attained owing to the presence of water vapor were given by Littwin.[3] A layer extending from 1000 to 700 mb, with a surface temperature of 24°C, dry-adiabatic lapse rate, and 100 per cent relative humidity realizes energy corresponding to a mean velocity of 12.8 m/sec by overturning. The calculation may be carried out either by the same method as that used by Margules or by the aid of a thermodynamic chart. In the latter case, it is important to take into account that during vertical readjustment in a layer of air of appreciable size both ascent and descent of air must take place, as already pointed out in Sec. 26.

Problem

29. Find the internal energy of a column of dry air with a constant lapse rate of temperature. Derive, also, an approximate expression for the internal energy of an air column of small height. The last expression involves only the surface pressure, the height of the air column, and, if terms of the second order are included, the surface temperature.

[1] Refsdal, A., *Geofys. Pub.*, **5**, No. 12, 1930.
[2] Raethjen, P., *Met. Z.*, **51**, 9, 1934.
[3] Littwin, W., *Monthly Weather Rev.*, **64**, 397, 1936.

CHAPTER XIII

THE GENERAL CIRCULATION OF THE ATMOSPHERE

93. Survey of the General Circulation. The circulation of the air at the earth's surface may briefly be described as follows: In the subtropical regions, easterly winds with components toward the equator prevail, the *NE trade winds* of the Northern Hemisphere and the *SE trade winds* of the Southern Hemisphere. They are separated near the equator by a zone of prevailing calms, the doldrums. Frequently, however, the doldrums are absent so that the two trade winds are separated by a surface of discontinuity[1] on which strong rain squalls may occur. Neither the two trade-wind regions nor the zone of doldrums are strictly symmetrical to the equator as can be seen from the following table which shows the position of the three regions during the two extreme months of the equinoxes, after Hann,[2] for the Atlantic and Pacific oceans:

Region of	March		September	
	Atlantic	Pacific	Atlantic	Pacific
NE trade...	26–3° north	25–5° north	35–11° north	30–10° north
Doldrum....	3° north–0°	5–3° north	11–3° north	10–7° north
SE trade....	0–25° south	3° north–28° south	3° north–25° south	7° north–20° south

The zone of calms is always north of the equator and the SE trades extend sometimes across the equator into the Northern Hemisphere. The circulation over the Indian Ocean, which is not included in this table, is completely modified by the monsoon circulation.

[1] Brooks, C. E. P., and Braby, H. W., *Quart. J. Roy. Met. Soc.*, **47**, 1, 1921. Durst, C. S., *Geophys. Mem.*, No. 28, 1926. Beals, E. A., *Monthly Weather Rev.*, **55**, 211, 1927.

[2] Hann-Süring, "Lehrbuch der Meteorologie," 4th ed., p. 469, Chr. Herm. Tauchnitz, Leipzig, 1926.

Beyond the trade-wind region is the zone of prevailing westerlies. They are better developed in the Southern than in the Northern Hemisphere, for in the latter the presence of the great continental surfaces causes many disturbances. The region of the westerlies is the region of the migrating cyclones and anticyclones; thus, instead of west winds, winds from other directions are frequently observed. The persistence of the west winds in these regions is much smaller than the persistence of the trade winds.[1] Poleward from the regions of westerlies, beyond latitudes of about 65 to 70°, winds with an easterly component prevail again.

The motion of the air is closely connected with the general pressure distribution.[2] Near the equator but, like the zone of doldrums, not quite symmetrically to it, a zone of low pressure is found, followed in both hemispheres by a zone of high pressure, the subtropical high-pressure belts. Especially in the summer hemisphere, these high-pressure belts do not extend continuously around the earth because high pressure cannot exist over the heated land. Consequently, the trade winds are also absent over land. During the winter in the Northern Hemisphere, a strong anticyclone persists over the Asiatic continent due to the very cold temperature in the surface layers.

The central regions of the high-pressure belts that coincide practically with the polar limits of the trade-wind regions frequently have calms.

In the region of the westerlies, even the mean pressure distribution is rather irregular owing to the migrating pressure centers. On the whole the pressure here decreases northward. In the surface layers of the arctic and antarctic regions the pressure distribution is anticyclonic owing to the low temperature of the surface layers. The zones of easterly winds in the polar regions are therefore not symmetric with respect to the poles, for the continental masses (Greenland and Antarctica) over which the greatest cooling of the surface layers occurs are not situated symmetrically to the poles.

[1] CONRAD, V., in W. Köppen-R. Geiger, "Handbuch der Klimatologie," Vol. 1*B*, p. 261, Gebrüder Bornträger, Berlin, 1936.

[2] Maps of the mean pressure distribution at the surface and aloft will be found in most textbooks on synoptic and on descriptive meteorology. See for instance N. Shaw, "Manual of Meteorology," Vol. 2, 2d ed., Cambridge University Press, London, 1934.

An orientation about the mean wind distribution aloft is best obtained by considering the mean pressure distribution at higher levels. Because the wind in the free atmosphere does not deviate much from the geostrophic wind, the wind field can be deduced approximately from the pressure field, with the aid of the equations of Sec. 53. The pressure field at higher levels

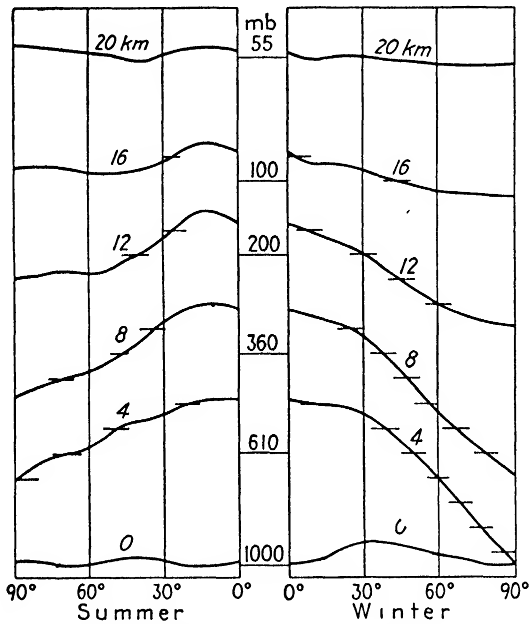

FIG. 67.—Pressure profiles at different altitudes (the short horizontal lines indicate pressure differences of 10 mb).

can be computed from the surface-pressure field and the vertical temperature distribution, as explained in Sec. 7. Charts of the upper pressure distribution have been constructed in this manner by Teisserenc de Bort and by Shaw.[1] The reader is referred to these maps for details.

Here a more summarizing representation is given in Fig. 67. This figure shows the pressure profiles in the meridional direction at six levels up to 20 km, after data given by Wagner.[2]

[1] SHAW, *loc. cit.*

[2] WAGNER, A., in W. Köppen-R. Geiger, "Handbuch der Klimatologie," Vol. 1*F*, p. 67.

The surface-pressure profile shows the features of the surface-pressure distribution described previously. Because the air throughout the troposphere is warmer in lower than in higher latitudes the vertical decrease of the pressure is smaller near the equator than near the poles. Therefore, the main characteristic of the pressure distribution at the upper level is a pressure decrease poleward. Already, at 4 km, the subtropical high and the slight polar high at the surface have vanished.

The meridional pressure gradients increase at first with the altitude. They are smaller in summer than in winter, for the meridional temperature gradients are smaller in summer than in winter. In the stratosphere the meridional pressure gradient must eventually become smaller again, for here the temperature increases northward (see Fig. 24). This effect probably becomes noticeable above 20 km.[1]

It follows from this pressure distribution that in summer the winds from the pole to the subtropics are predominantly westerly and very light in the tropics where the inclination of the isobaric surfaces is small. Only in tropical regions will easterly winds be found. In winter the winds are mainly westerly over the whole hemisphere. At high levels above 20 km, where the effect of the stratospheric temperature increase toward the pole makes itself felt, easterly winds are to be expected; but, in the present state of our knowledge, this is merely hypothetical.

An inspection of upper-air pressure charts reveals that the pressure distribution shown in Fig. 67 and the wind distribution deduced from it are disturbed considerably by the effect of the continents. Wind observations from pilot-balloon ascents are not yet numerous enough to give a complete picture of the circulation everywhere in the earth's atmosphere. But the observations so far bear out the deductions made from the pressure distribution.

No cross section showing a scheme of the general circulation over the whole earth is given here, for it cannot be emphasized too strongly that the general circulation is *not* the same everywhere around one and the same parallel. The deviations from the zonal symmetry can be explained by the distribution of water and land. But it will be seen in the following sections that these

[1] Stüve, G., "Handbuch der Geophysik," Vol. 9 (2), p. 443, Gebrüder Bornträger, Berlin, 1937.

deviations from zonal symmetry are a necessary condition for the exchange of heat between the warm tropical zones and the cold polar regions. The asymmetries would also develop on a rotating globe with a completely uniform surface. Only the location of the asymmetries, not their existence, is determined by the distribution of continents and oceans.

94. Application of the Circulation Theorem to the General Circulation. A satisfactory theory of the general circulation should explain the air motions outlined in the preceding section on the basis of the known incoming solar radiation which represents the source of energy, the known distribution of land and water, and the known properties of the air. At present, it cannot be claimed that this problem is even partly solved. Therefore, in the following discussion, only certain theoretical considerations will be presented that have contributed or may contribute in the future to a more thorough understanding of the mechanism of the general circulation.

For these reasons, older investigations by Ferrel, Siemens, Overbeck, and others[1] will be omitted. They are very important for the development of dynamic meteorology as a whole, for they represent some of the first attempts to apply dynamic principles to meteorology. But, as far as their direct applications to the general circulation is concerned, they are only of historical interest.

On a nonrotating earth with a homogeneous surface heated at the equator and cooled at the poles the general circulation would consist simply of a motion from the pole to the equator in the lower atmosphere, an ascent of the warmer air in the equatorial regions, a return to the pole in the upper layers, and a descent of the cooled air in polar regions. Thus, two vortexes would exist, one in the Northern, the other in the Southern Hemisphere. The mean intensity of the circulation can be calculated from the circulation theorem in the form

$$\frac{dC}{dt} = R(\overline{T_a} - \overline{T_b}) \ln \frac{p_0}{p_1} - 2\omega \frac{dF}{dt} \tag{52.4}$$

[1] Brillouin, M., "Mémoires originaux sur la circulation générale de l'atmosphère," G. Carré et C. Naud, Paris, 1900. Abbe, C., "The Mechanics of the Earth's Atmosphere," 2d collection, 1891, 3d collection, 1910, Smithsonian Institution, Washington, D. C.

The second term on the right-hand side of this equation represents the effect of the earth's rotation which will be considered later. To study the intensity of the general circulation on a non-rotating earth, a path of integration may be chosen along the isobars 1000 and 300 mb, following Bjerknes.[1] They coincide approximately with the surface of the earth and the cirrus level (9000 m). The two verticals with the mean temperatures $\overline{T_a}$ and $\overline{T_b}$ may be situated at the equator and at the pole (Fig. 68). As a representative value for the difference of these mean temperatures, Bjerknes chooses

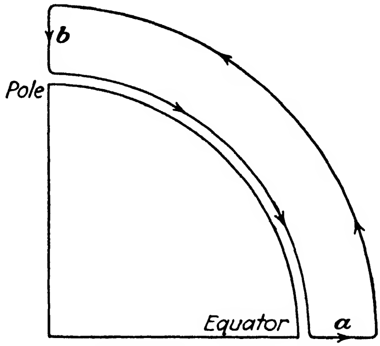

FIG. 68.—Circulation on a non-rotating earth.

$$\overline{T_a} - \overline{T_b} = 40°\text{C}$$

It follows that

$$\frac{dC}{dt} = 138 \times 10^6 \text{ cm}^2/\text{sec}^2$$

Because the length of the vertical branches of the curve can be neglected, the length of the path is equal to half the circumference of the earth, 2×10^9 cm. According to (52.1) the mean meridional acceleration

$$\overline{\frac{dV}{dt}} = 6.9 \times 10^{-2} \text{ cm/sec}^2$$

The resulting mean velocities after a given time are shown in the second line of the table on page 260. These velocities as well as the quantities to be discussed later are computed under the assumption that the mean acceleration remains constant. The circulation would attain a considerable velocity after 6 hr and would have reached hurricane intensity after 24 hr. The third line of the table gives the mean meridional displacement. If the original acceleration is maintained, the air would have moved

[1] BJERKNES, V., and collaborators, "Physikalische Hydrodynamik," p. 645, Verlag Julius Springer, Berlin, 1933.

one-fourth of the way from the pole to the equator in a day and would describe the whole closed path from the pole to the equator and back in 67 hr.

THEORETICAL VELOCITY OF THE GENERAL CIRCULATION

Time	1 sec	3 hr	6 hr	24 hr
Mean Meridional velocity, m/sec......	6.9×10^{-4}	7.4	14 9	59.6
Mean Meridional displacement, km....	0.345×10^{-6}	40.4	160.5	2580
Mean zonal velocity at 30° lat, m/sec......	2.52×10^{-8}	2.94	11.7	188
Counteracceleration at 30° lat, cm/sec²....	1.84×10^{-10}	2.14×10^{-2}	8.51×10^{-2}	1.37

It must, however, be emphasized that even apart from the effect of the earth's rotation, which will be discussed presently, the motion could not be maintained at the rate shown. The acceleration depends on the temperature difference between the pole and the equator. When a transport of air takes place between the pole and the equator, the temperature difference is reduced if it is not reestablished by heating at the equator and cooling at the pole. As long as the motion is not too fast, the heating and cooling effects will be capable of maintaining a temperature difference. As the circulation becomes faster, the temperature difference is diminished. Therefore, the acceleration in its turn decreases. An equilibrium state will be reached in which the heat transport, owing to the circulation between pole and equator, just balances the effect of heating and cooling at low and high latitudes so that the temperature difference remains constant. The acceleration due to this temperature difference will then be used, not to speed up the motion, but to overcome the effects of friction which has been left out of consideration so far. Thus, a stationary circulation could develop even on a nonrotating earth.

The existence of such a simple meridional circulation is, however, impossible owing to the deflective force of the earth's rotation. Both currents, toward the equator in the lower atmosphere and toward the pole in the upper atmosphere, are subjected to deviations, toward the right in the Northern Hemisphere.

Instead of deriving the magnitude of these deviations by means of the second term on the right-hand side of (52.4) the equations of motion (47.2) will be used directly. The motion considered so far is, of course, not a motion under balanced forces for the Coriolis force does not come into play. The air moves rather in the direction of the pressure gradient, from north to south in the lower atmosphere and from south to north in the upper atmosphere. If v is the velocity component in meridional direction, which has alone been considered hitherto, it follows from the first equation (47.2), because the direction of the pressure gradient is meridional, that the acceleration in longitudinal direction is given by

$$\frac{du}{dt} = 2\omega \sin \varphi \, v = 2\omega \sin \varphi \frac{dy}{dt} \tag{94.1}$$

By integration,

$$u = 2\omega \sin \varphi (y - y_0) \tag{94.2}$$

when the variation of the Coriolis parameter is neglected. Equation (94.2) shows the west to east velocity which results from a displacement in meridional direction. When the displacement is northward, the velocity is directed toward the east on the Northern Hemisphere; when the displacement is southward, the velocity is toward the west. The zonal velocities caused by the meridional displacements after certain times are shown in the third line of the table on page 260. These velocities are directed toward the east in the upper part and toward the west in the lower part of the circulation. After 6 hr the zonal velocity is already almost as large as the meridional velocity, and after 24 hr it would be three times larger. Equation (94.2) is, of course, not strictly correct, for the latitude varies during the meridional displacement. Nevertheless, the importance of the Coriolis effect is clearly brought out.

The zonal velocity component expressed by (94.2) produces a Coriolis acceleration $2\omega \sin \varphi \, u$ which is opposite to the original circulation acceleration, southward in the higher atmosphere, northward in the lower atmosphere. The numerical values of this counteracceleration are given in the fourth line of the table on page 260. It will be seen that after less than 6 hr the counter-acceleration is greater than the original acceleration (6.9×10^{-2} cm/sec^2) caused by the temperature difference between high and

low latitudes, so that the general circulation should be mainly parallel to the circles of latitude.

This discussion of the general circulation based on the circulation theorem is, of course, very schematic. The Coriolis force begins to deflect the meridional motion as soon as the latter starts. The point of the preceding calculations is that the atmosphere on the rotating earth cannot circulate in a prevailing meridional direction but must have a strong zonal component owing to the effect of the earth's rotation. This strong zonal component of the motion requires that the meridional component of the pressure gradient be strong, according to the geostrophic wind relation which is satisfied with a high degree of approximation.

95. The Meridional Heat Transport. A transport of heat from equatorial to polar regions must take place, for the temperature difference between pole and equator remains constant—apart from seasonal and other smaller deviations from the mean value—in spite of the greater amount of heat received at the equator than at the pole. The necessity of a heat transport from the equator to the pole is directly shown by the table on page 102. At the latitudes below about 35°, the incoming radiation is greater than the outgoing; at higher latitudes the reverse is true. The surplus of heat in the zone from 0 to 10° of latitude must be carried across the parallel 10° latitude, the surplus of heat in the zone 0 to 20° latitude must be carried across the parallel 20° latitude, and so on. Up to about 35° latitude the heat transport increases, therefore; at higher latitudes, it decreases because the incoming radiation is here less than the outgoing so that part of the heat is retained in each latitudinal belt to make up the deficit.

The total surplus or deficit for each zone can be obtained from the figures given in the table on page 102. The difference between incoming and outgoing radiation represents the surplus or deficit per unit area. It has to be multiplied by the area of the latitudinal belt in order to obtain the surplus or deficit for the whole zone. The sum of the surpluses and deficits for all zones of a hemisphere should vanish. Actually, when Albrecht's figures (page 102) are used, a net deficit of heat is found. Bjerknes[1] has therefore reduced Albrecht's figures for the outgoing radiation by about 1 to 2 per cent, so that the net deficit of energy vanishes.

[1] BJERKNES, V., and collaborators, *op. cit.*, p. 665, Berlin 1933.

Bjerknes's figures are given in the following table, but they have been changed into thermal units. The first three columns show

MERIDIONAL HEAT FLOW AFTER ALBRECHT AND BJERKNES

Latitude, deg	Incoming radiation, cal/cm² sec	Outgoing radiation, cal/cm² sec	Difference, cal/cm² sec	Surplus per 10° zone, cal/sec	Heat flow, cal/sec
0	564.8 $\times 10^{-6}$	499.8 $\times 10^{-6}$	65.0 $\times 10^{-6}$	26.49 $\times 10^{13}$	0 $\times 10^{13}$
10	555.7	497.9	57.8	22.48	26.49
20	532.7	489.7	43.0	13.00	48.97
30	494.8	471.8	23.0	3.99	61.97
40	444.8	452.7	− 7.9	− 9.60	65.96
50	386.8	429.8	− 43.0	−16.48	56.36
60	320.8	407.8	− 87.0	−19.90	39.88
70	266.9	384.9	−118.0	−14.96	19.88
80	239.9	367.0	−127.1	− 5.29	5.29
90	233.9	367.0	−133.1		0.0

the incoming and outgoing radiation and the difference between the two. The fourth column gives the total surplus or deficit for zones of 10° latitude. The last column contains the flow of heat across each tenth parallel. The figures are, of course, not very reliable, for the data on which they are based are known only for a small part of the earth. Other investigators, as mentioned in Sec. 37, have obtained somewhat different results. Nevertheless, the above values may be considered as sufficient for a first orientation.

A small fraction of this heat flow may be carried by the ocean currents. Furthermore, water vapor evaporated in the tropics and transferred poleward brings energy in the form of latent heat of condensation to higher latitudes which is released again when condensation occurs. This form of energy transport seems to be especially effective in the trade-wind region, according to von Ficker.[1] The main part of the heat flow must, however, be due to the direct exchange of air between higher and lower latitudes.

It was shown in the preceding section that no uniform meridional circulation can exist on the rotating earth but that the motion must rather be zonal, especially at middle latitudes, where the deflection of air coming from equatorial and polar regions would reach high values. In conjunction with this zonal

[1] VON FICKER, H., *Sitz.-Ber. preuss. Akad. Wiss. Phys.-Math. Kl.*, **11**, **103, 1936.**

motion a north to south pressure gradient should exist, according to the geostrophic wind relation. In order to make possible the meridional wind component required for the heat transport, east to west pressure gradients must exist, as pointed out by Exner.[1] The appearance of such a pressure gradient $\partial p/\partial x$ in (94.1) counteracts the Coriolis force and therefore either annuls du/dt or makes it so small that the motion can have an appreciable meridional component even at middle latitudes. However, such a zonal component of the pressure gradient cannot have the same direction all around the earth, for the pressure must be continuous around the circle of latitude. The pressure gradient must therefore be different across different meridians, and the general circulation which satisfies approximately the gradient-wind relation must be broken up into a number of smaller circulations which lie side by side along the meridians, as is shown schematically in Fig. 69. A circulation of a similar type is what is actually seen in the cyclones and anticyclones. The number of cyclones and anticyclones indicated in Fig. 69 is, of course, arbitrary, and the symmetrical distribution does not correspond to what is observed in nature. On a completely homogeneous earth the distribution of the cold and warm currents would be accidental, although presumably symmetrical. Owing to the differences on the surface of the earth, however, the location of the cold and warm currents is largely determined by the position of land and sea. Consequently the distribution of the large centers of high and low pressures is also fixed, as manifested by the semipermanent centers of action such as the Aleutian and Icelandic lows or the Pacific high.

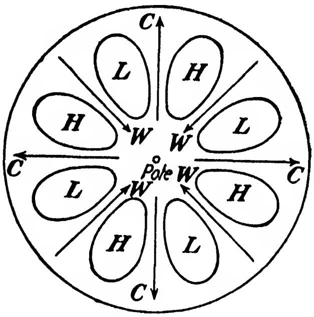

FIG. 69.—Scheme of the general circulation between lower and higher latitudes; H = high, L = low, W = warm current, C = cold current. (*After Exner.*)

It appears, therefore, that for a thorough study of the general circulation not much is gained by assuming as a "first approximation" that the surface of the earth is uniform, because the

[1] EXNER, F. M., "Dynamische Meteorologie," 2d ed., pp. 215–218, Verlag Julius Springer, Vienna, 1925.

distribution of water and land gives a valuable clue to the distribution of warm and cold currents.

Jeffreys[1] arrived at the same conclusion as Exner, *viz.*, that the general circulation cannot be symmetrical around the pole, by means of a different argument. The atmosphere must continually lose or gain momentum by friction. If the momentum in the direction of the earth's rotation is counted positive, air moving eastward—faster than the earth's surface—loses momentum, whereas the air moving westward gains momentum by friction. In a steady state, these gains and losses must be compensated. The only effective process that can bring about compensation of the gains and losses is the interchange of momentum with air coming from other latitudes. The interchange of momentum depends on the velocity in the meridional direction. With purely zonal isobars, a meridional velocity component can exist only in the lowest layers of the atmosphere, owing to friction, as follows from Sec. 76. Jeffreys has shown that this component is much too feeble to produce an interchange of momentum sufficient to replace the losses of momentum by friction. If the pressure distribution is not purely zonal, the meridional component of motion can be much stronger and thus it is possible to compensate for the frictional losses of momentum by transfer of momentum from other regions where momentum is gained.

96. The Meridional Heat Transport as a Form of Turbulent Mass Exchange. It is possible to consider the meridional transport of heat as the effect of a horizontal turbulent mass exchange in the direction from the equator to the pole as has been shown by Defant[2] and others.[3] The general circulation of the atmosphere may then be regarded as a turbulence phenomenon on a very large scale. Though in the ordinary problems of turbulence the eddies responsible for the mixing of air of different properties are small, their dimensions being measured in meters, the migrating cyclones and anticyclones must be regarded as

[1] Jeffreys, H., *Un Géodés. Géophys. Int.*, 5 ème assemblée gén. Lisb., 1933; *Proc.-Verb. de l'Ass. de Mét.*, II, p. 219.

[2] Defant, A., *Geografiska Annaler*, **3**, 209, 1921; *Sitz.-Ber. Akad. Wiss. Wien.*, **130**, 383, 1921.

[3] Exner, *op. cit.*, p. 239. Ångström, A., *Arkiv Mat., Astron. Fysik*, **19**, *A*, No. 20, 1925. Lettau, H., *Gerl. Beitr. Geophys.*, **40**, 390, 1931; *Ann. Hydr.*, **62**, 152, 1934; *Beitr. Phys. Atm.*, **23**, 45, 1935.

the turbulent eddies of the general circulation. Obviously, the analogy is rather crude, for one of the important points of the ordinary theory of turbulence is that the number of turbulent eddies in the current is so large that one is justified in speaking of mean conditions. The migrating cyclones and anticyclones are, of course, not numerous enough to determine a statistical mean eddy. Nevertheless, the analogy is useful, for it permits the computation of a coefficient of turbulent mass exchange for the meridional heat transport that gives a numerical expression for the intensity of the general circulation.

The expression for the transport of heat S in the meridional direction,

$$S = -c_p \frac{A}{E} \frac{dT}{d\varphi} \qquad (96.1)$$

by analogy with (81.41). The specific heat of air at constant pressure is c_p, the coefficient of turbulent mass exchange is A, the temperature is T, and the earth's radius is E, and the latitude is φ. It is permissible to use the temperature here instead of the potential temperature, for the variation of the pressure in the horizontal direction is sufficiently small not to affect the following estimates.

When the meridional heat transport S and the meridional temperature gradient for a given latitude are known, the coefficient of turbulent mass exchange A for this latitude can be found. According to the table on page 263, the heat flow across 40° latitude is 66×10^{13} cal/sec. This amount of heat passes per second through a vertical surface erected on the parallel of 40° latitude. To find the heat flow per square centimeter through this surface it may be assumed that its height is 5 km. An accurate value of the height is not needed, for only the order of magnitude of A is to be determined. Because the circumference of the parallel 40° is 3.07×10^9 cm, it follows that $S = 0.430$ cal/cm² sec. The meridional temperature gradient can be assumed as 5°C/1000 km or 5×10^{-8} °C/cm, according to Fig. 24. Because $c_p = 0.24$, $A = 4 \times 10^7$ gm/cm sec, approximately. This value of A is 10^5 to 10^6 times larger than the figures given in Chap. XI for the coefficient of turbulent mass exchange in the vertical direction and at least 100 times larger than the coefficient of lateral mixing (Sec. 85), for the dimensions

of the "turbulent eddies" of the general circulation are much greater.

This value of A represents, of course, only the order of magnitude; for the meridional temperature distribution depends not only on the atmospheric but also on the oceanic circulation, and heat is also transported with the water vapor as latent heat of condensation. Nevertheless, our estimate agrees well with the results obtained by other authors and may be considered reliable as far as the order of magnitude is concerned.

Defant has also determined A directly by means of Eq. (81.5)

$$A = \frac{\Sigma ml}{Ft} \tag{81.5}$$

Such a method is practicable in the present instance when each "turbulent element," *i.e.*, each cyclone and anticyclone, can be observed directly. The order of magnitude of A is the same as previously.

The coefficient of turbulent mass exchange of the general circulation changes with the latitude. Lettau[1] has given the following figures from an investigation based on the observations in the Northern Hemisphere during the polar year 1932 to 1933:

Latitude	35°	40°	45°	50°	55°	60°	65°	70°
A, gm/cm sec	4.4	6.2	7.0	8.2	8.2	7.0	7.1	6.8×10^7

97. The Cellular Structure of the General Circulation. It has already been pointed out that the general circulation of the atmosphere must be broken up into a number of smaller circulations in order to permit a heat transport between lower and higher latitudes, as shown in Fig. 69, after Exner. The different circulation cells cannot extend all the way from the pole to the equator, for the air would then acquire very large zonal velocities owing to the conservation of angular momentum.

V. Bjerknes and his collaborators[2] have given a model of a cellular subdivision of the atmosphere that does not contradict the theoretical considerations restricting the air motion over the globe and that is in agreement with the observations.

[1] LETTAU, H., *Beitr. Phys. Atm.*, **23**, 45, 1936.
[2] BJERKNES, V., and collaborators, *op. cit.*, p. 680.

Their scheme is partly reproduced in Fig. 70. The zonal section at 30° north latitude (Fig. 70*a*) shows a cell whose width has been assumed arbitrarily 90° in longitude. The particles move around their orbits in anticyclonic sense, ascending while they are in the tropical easterly current and descending in the westerly current north of 30° latitude. The orbital planes are therefore ascending toward the west. It follows that the air having completed the southern, ascending branch of its anticyclonic orbit

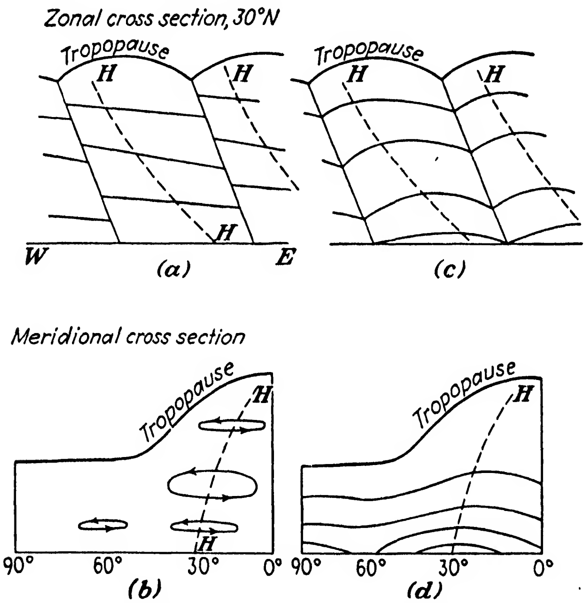

FIG. 70.—The cells of the general circulation. (*a*), (*b*) Orbits of the particles; (*c*), (*d*) isobars. (*After Bjerknes.*)

has been lifted so that, according to Sec. 27, it has become less stable. Consequently, precipitation may be expected here. This agrees with the observations. On the whole the subtropical high-pressure belt is very dry, but, at the western end of each anticyclonic cell, showers occur frequently.

The inclination of the orbital planes may be about 1:10,000, so that the vertical amplitudes owing to the orbital motion may reach 1 km. The orbital planes are inclined most in the middle troposphere, at about 5 km altitude, and less near the surface of the earth and near the tropopause. The tropopause represents

the upper boundary of the cells. Therefore, the uppermost air in the troposphere must move tangentially to the tropopause, and the latter obtains the form shown in Fig. 70*a*.

Figure 70*c* shows the isobars corresponding to the motion of the air in the cells provided that the wind does not deviate much from the gradient-wind motion. The axis of the anticyclone is not quite vertical but is inclined toward the west. The highest surface pressure is displaced to the east of the center of the cell, as indicated by the observations.

The boundaries between the different cells are surfaces of discontinuity between different air masses whose inclination can be calculated from the formulas of Chap. VIII. A sufficient number of aerological observations to indicate the existence of these cell walls have been made only in the North Atlantic west of the African coast where the cell wall forms the boundary between the NE trade winds on the easterly end of the Azores and the upper SW antitrade winds on the westerly end of the high at greater altitudes over Africa. The trade-wind circulation in this region has been studied very thoroughly by Sverdrup[1] and by von Ficker.[2] Both found that the loss of heat is mainly due to radiation and that heat is added mainly in the form of latent heat of evaporation from the ocean.

The meridional projection of the orbital paths that the particles describe in these anticyclonic cells is shown in Fig. 70*b*. The direction of the meridional component of the motion is the same as was deduced in Sec. 94 for the circulation on a nonrotating earth that is heated in equatorial and cooled in polar regions. The position of the isobars in the meridional cross section is shown in Fig. 70*d*.

This simple picture of the anticyclonic cells of the lower latitudes requires some modifications if the cells are not stationary but moving eastward. The orbital paths of the particles are then trochoidal curves. But the direction of the meridional component of the circulation that takes care of the heat transport and is driven by the temperature difference between low and high latitudes remains unchanged by the superposition of a translatory motion.

[1] SVERDRUP, H. U., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **2**, 1, 1917.

[2] FICKER, H. VON., *Veroffentlich. Met. Inst. Univ. Berlin*, Vol. 1, No. 4, 1936; *Sitz.-Ber. preuss. Akad. Wiss. Phys.-Math. Kl.*, **11**, 103, 1936.

Similar considerations may be applied to the stationary lows at about 60° latitude, such as the Icelandic and Aleutian lows; but a comparison with the observations is here not possible, for in the mean-pressure charts not only the effects of these stationary lows but also those of the migrating cyclones of the polar and arctic fronts make themselves felt. Closed orbits even in the stationary lows are at these latitudes found only in the lower troposphere. Owing to the meridional temperature gradient the current in the upper troposphere is mainly from west to east with superimposed sinusoidal oscillations in the meridional direction (see page 166).

On a planet with a homogeneous surface the position of the various cells of the general circulation would be arbitrary. On the earth, however, with its surface varying between water and land the position of the cells is determined largely by geographical factors.

CHAPTER XIV

THE PERTURBATION THEORY OF ATMOSPHERIC MOTIONS

98. Disturbed and Undisturbed Motion. The mathematical solution of most problems of dynamic meteorology is exceedingly difficult owing to the many factors that have to be taken into account. In classical hydrodynamics the fluids that are investigated are as a rule considered as incompressible and homogeneous, *i.e.*, of the same density everywhere. Moreover, the effects of the earth's rotation are mostly neglected. In the atmosphere, however, these factors are of importance and have to be taken into consideration.

Even the mathematical study of an incompressible, homogeneous fluid on a nonrotating earth presents great difficulties. This is due to the fact that in the equations of motion (47.2), terms of the second degree appear, *viz.*, $u\,\dfrac{\partial u}{\partial x}, \ . \ . \ . \ , w\,\dfrac{\partial w}{\partial z}$. The solution of such second degree differential equations is very difficult, but linear differential equations can be handled comparatively simply according to standard methods,

In many problems, it is fortunately possible to reduce the hydrodynamic equations to a linear form. The motion that is to be studied can often be treated as a small perturbation superimposed on an undisturbed state of the atmosphere. Such problems arise, for instance, in the theory of the origin of extratropical cyclones. According to the theory of V. Bjerknes and his collaborators, cyclones develop as small wave perturbations at the boundary between two air masses of different density and velocity, as will be discussed in detail later (Sec. 109). If such a wave is unstable, its amplitude increases and a cyclone develops. Thus, the first mathematical problem to be solved in connection with the development of cyclones is that of the stability of waves at a surface of discontinuity. These waves have, at the beginning, small amplitudes so that the wave motion may be regarded as a small perturbation superimposed on a comparatively simple

undisturbed motion. In the case of the cyclone problem, it may be assumed that the undisturbed motion is geostrophic. V. Bjerknes[1] has shown how the hydrodynamic equations can be brought into a linear form when such perturbations are studied. The method of linearization of the hydrodynamic equations had been used prior to Bjerknes, but Bjerknes first developed the method systematically for both compressible and incompressible fluids. In the following sections, some relatively simple examples of the perturbation theory will be given which will enable the reader to understand, among other things, the background of the wave theory of cyclones. For a complete exposition of the perturbation theory and its results the reader is referred to "Physikalische Hydrodynamik," by V. Bjerknes and his collaborators.

99. The Perturbation Equations. In deriving the perturbation equations, the following assumptions are made:

1. Not only the total disturbed plus undisturbed motion satisfies the hydrodynamic equations but also the undisturbed motion alone.

2. The perturbations are so small that terms of the second order in the perturbation quantities can be neglected with respect to terms of the first order in the perturbation quantities.

In order to avoid too lengthy calculations, we shall furthermore assume, in all examples, that the fluid is incompressible. It will be convenient throughout this chapter to write for the x-, y-, and z-components of the vector of the earth's rotation Ω_x, Ω_y, and Ω_z. In a coordinate system whose x-axis makes an angle β with the east direction,

$$\Omega_x = \omega \cos \varphi \sin \beta, \qquad \Omega_y = \omega \cos \varphi \cos \beta, \qquad \Omega_z = \omega \sin \varphi \quad (99.1)$$

according to the footnote on page 125.

The variables of the undisturbed motion will henceforth be indicated by capital letters, the variables of the disturbed motion by small letters, and the variables of the total motion by small letters with a bar. Then, according to the principle of superposition,

$$\bar{u} = U + u, \qquad \bar{v} = V + v, \qquad \bar{w} = W + w, \qquad \bar{p} = P + p \quad (99.2)$$

[1] Bjerknes, V., *Beitr. Phys. Atm.*, **13**, 1926; *Geofys. Pub.*, **5**, No. 11, 1929.

Because the fluid is incompressible, the density is the same in the disturbed and in the undisturbed motion. These quantities satisfy the three equations of motion (47.2)

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + 2\Omega_y \bar{w} - 2\Omega_z \bar{v} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} \\
\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z} + 2\Omega_z \bar{u} - 2\Omega_x \bar{w} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} \qquad (99.3) \\
\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z} + 2\Omega_x \bar{v} - 2\Omega_y \bar{u} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} - g
\end{aligned}
$$

and the equation of continuity (47.3) for an incompressible fluid

$$
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (99.4)
$$

According to assumption 1 the same equations (99.3) and (99.4) are satisfied by the undisturbed motion alone. It is not necessary to write these equations down explicitly, for they are obtained from (99.3) and (99.4) by introducing capital letters for the variables.

To find the perturbation equations, the expressions (99.2) have to be inserted in (99.3) and (99.4). Thus, it follows, for instance, for the third equation of motion (99.3) that

$$
\begin{aligned}
&\underline{\frac{\partial W}{\partial t}} + \frac{\partial w}{\partial t} + \underline{U\frac{\partial W}{\partial x}} + u\frac{\partial W}{\partial x} + U\frac{\partial w}{\partial x} + \underline{\underline{u\frac{\partial w}{\partial x}}} + \underline{V\frac{\partial W}{\partial y}} + v\frac{\partial W}{\partial y} \\
&\quad + V\frac{\partial w}{\partial y} + \underline{\underline{v\frac{\partial w}{\partial y}}} + \underline{W\frac{\partial W}{\partial z}} + w\frac{\partial W}{\partial z} + W\frac{\partial w}{\partial z} + \underline{\underline{w\frac{\partial w}{\partial z}}} + \underline{2\Omega_x V} \\
&\qquad + 2\Omega_x v - \underline{2\Omega_y U} - 2\Omega_y u = -\underline{\frac{1}{\rho}\frac{\partial P}{\partial z}} - \frac{1}{\rho}\frac{\partial p}{\partial z} - \underline{g}
\end{aligned}
$$

In this rather lengthy equation the terms that are underlined once satisfy by themselves the equation for the undisturbed motion and cancel each other. The terms that are underlined twice are of the second degree in the perturbation quantities and may be neglected. Upon applying the same reasoning to the other two equations of motion, it follows that

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial z} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + w\frac{\partial U}{\partial z} \\
&\qquad\qquad + 2\Omega_y w - 2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x}
\end{aligned}
$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial z} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z} + 2\Omega_z u - 2\Omega_x w = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (99.5)$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + V\frac{\partial w}{\partial y} + W\frac{\partial w}{\partial z} + u\frac{\partial W}{\partial x} + v\frac{\partial W}{\partial y} + w\frac{\partial W}{\partial z} + 2\Omega_x v - 2\Omega_y u = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Similarly, the equation of continuity can be written when the terms are omitted that satisfy the equation of continuity for the undisturbed motion,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (99.6)$$

At a first glance, it might appear that the perturbation equations (99.5) and (99.6) are more complicated than the original equations. However, it should be remembered that in these equations the undisturbed velocity may be regarded as known. The system of Eqs. (99.5), and (99.6) is actually simpler than the system (99.3) and (99.4) because the former contains only linear terms. Furthermore, when we consider special examples of perturbations superimposed on undisturbed motions, many of the terms in (99.5) and (99.6) will drop out, as will be seen in the later sections of this chapter.

100. The Boundary Conditions. To complete the system of perturbation equations the boundary conditions have to be introduced. A boundary of the fluid may be given by the equation

$$\mathfrak{f}(x, y, z, t) = 0 \quad (100.1)$$

The boundary must always be formed by the same particles; otherwise, it would dissolve. If the coordinates of a particle in the boundary after a very short time interval dt are $x + dx$, $y + dy$, $z + dz$, the new coordinates must satisfy the same equation. Therefore,

$$\mathfrak{f}(x + dx, y + dy, z + dz, t + dt) = 0$$

or, developing into a Taylor series,

$$\mathfrak{f}(x, y, z, t) + \frac{\partial \mathfrak{f}}{\partial x}dx + \frac{\partial \mathfrak{f}}{\partial y}dy + \frac{\partial \mathfrak{f}}{\partial z}dz + \frac{\partial \mathfrak{f}}{\partial t}dt = 0$$

Since dx/dt, dy/dt, dz/dt are the total velocities, disturbed and undisturbed, it follows in view of (100.1) that

$$\frac{\partial \bar{f}}{\partial t} + \bar{u}\frac{\partial \bar{f}}{\partial x} + \bar{v}\frac{\partial \bar{f}}{\partial y} + \bar{w}\frac{\partial \bar{f}}{\partial z} = 0 \tag{100.2}$$

If the boundary surface is an internal surface between two fluid masses of different density and velocity, two equations of the form (100.2) must be satisfied at the surface of discontinuity. They are obtained by substituting for $\bar{u}$, $\bar{v}$, and $\bar{w}$ the velocity components at the boundary first in one, then in the other layer.

In many cases the equation of the boundary surface will be given explicitly; in others, it may have to be determined by dynamical considerations. Thus, the pressure at the boundary surface must be the same on both sides [see (60.1)] of the boundary. If the pressure on one side of the boundary is $\bar{p}$ and on the other side $\bar{p}'$,

$$\bar{p} - \bar{p}' = 0 \tag{100.3}$$

When the boundary surface is a free surface bounded by a region with constant pressure, the dynamic condition may be written

$$\bar{p} = \text{const} \tag{100.31}$$

In order to fit the boundary condition (100.2) into the system of perturbation equations, the equation for the undisturbed boundary may be written

$$F(x, y, z, t) = 0 \tag{100.4}$$

while

$$\bar{f} \equiv F + f \tag{100.5}$$

Thus f represents the effect of the perturbation on the position of the boundary surface.

Because the undisturbed motion must satisfy the undisturbed boundary condition, it follows that

$$\frac{\partial F}{\partial t} + U\frac{\partial F}{\partial x} + V\frac{\partial F}{\partial y} + W\frac{\partial F}{\partial z} = 0 \tag{100.6}$$

Upon substituting from (99.2) and from (100.5), the boundary condition (100.2) becomes, if terms of higher order are neglected,

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} + u\frac{\partial F}{\partial x} + V\frac{\partial f}{\partial y} + v\frac{\partial F}{\partial y} + W\frac{\partial f}{\partial z} + w\frac{\partial F}{\partial z} = 0 \tag{100.7}$$

When the equation for the boundary surface in the form (100.3) is to be used, f must be replaced by the difference of the perturbation pressure $p - p'$ and F by the difference of the undisturbed pressures $P - P'$.

$$\frac{\partial(p - p')}{\partial t} + U\frac{\partial(p - p')}{\partial x} + u\frac{\partial(P - P')}{\partial x} + V\frac{\partial(p - p')}{\partial y} + v\frac{\partial(P - P')}{\partial y} + W\frac{\partial(p - p')}{\partial z} + w\frac{\partial(P - P')}{\partial z} = 0 \quad (100.8)$$

Some elementary examples of the perturbation method will be given in the following sections of this chapter.

A further approach to reality would be obtained by including the frictional terms in the perturbation equations, but for many important meteorological problems the effects of friction appear to be of secondary importance.

101. Wave Motion at the Free Surface of a Single Layer. Gravitational Waves. As a first example we shall consider the wave motion in a single layer. The effect of the earth's rotation may be neglected. The bottom of the layer may be a rigid surface at the height $z = 0$; the upper free surface is a horizontal plane in the undisturbed case, for the isobaric surfaces are horizontal if the effect of the earth's rotation is disregarded. Thus, the equation for the free surface in the undisturbed motion has the form

$$z - h = 0 \quad (101.1)$$

where h is the depth of the layer.

In the undisturbed case the fluid may move with the constant velocity U along the x-axis. It should be noted that this is not a geostrophic motion, for the earth's rotation is neglected. The fluid system represents, rather, water flowing in a channel of very great width. Under these conditions, only the third of Eqs. (99.3) remains for the undisturbed motion.

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \quad (101.2)$$

It may be assumed that the motion does not depend on the y-coordinate, and thus the perturbation equation (99.5) for the y-component can be omitted. The motion is two-dimensional in the vertical xz-plane. At all points with the same x- and

z-coordinates the velocity and the pressure are the same even though y is different.

With these simplifications the perturbation equations (99.5) are reduced to

$$\begin{aligned} \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) u &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \end{aligned} \tag{101.3}$$

Since U and ρ are assumed to be constant, this system of linear differential equations has constant coefficients. The solutions are therefore exponential or trigonometric functions. In order to study wave perturbations in the x-direction, the following form of the solution may be assumed:

$$\begin{aligned} u &= A \cos\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} \\ w &= C \sin\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} \\ p &= D \cos\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} \end{aligned} \tag{101.31}$$

A, C, and D are constants on which the amplitudes of u, w and p depend. The constant λ is the wave length, and c the wave velocity. The constant γ determines the variation of the amplitudes with the height. When the expressions (101.31) are substituted in (101.3), one obtains the result

$$+A\frac{2\pi}{\lambda}(c - U)\sin\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} = +\frac{D}{\rho}\frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}(x - ct)e^{\gamma z}$$

$$-C\frac{2\pi}{\lambda}(c - U)\cos\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} = -\frac{D}{\rho}\gamma\cos\frac{2\pi}{\lambda}(x - ct)e^{\gamma z}$$

$$-A\frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} + \lambda C\sin\frac{2\pi}{\lambda}(x - ct)e^{\gamma z} = 0$$

Division of the first and third of these equations by

$$\sin\frac{2\pi}{\lambda}(x - ct)e^{\gamma z}$$

and of the second equation by $\cos\frac{2\pi}{\lambda}(x - ct)e^{\gamma z}$ shows that the

expressions (101.31) satisfy the differential equations (101.3) provided that

$$
\begin{aligned}
A\frac{2\pi}{\lambda}(c - U) &= \frac{2\pi}{\lambda}\frac{D}{\rho} \\
C\frac{2\pi}{\lambda}(c - U) &= \gamma\frac{D}{\rho} \\
A\frac{2\pi}{\lambda} \qquad\qquad &= \gamma C
\end{aligned}
$$

We shall express A and D by C, the constant factor of the vertical velocity. From the third equation, it follows that

$$A = \frac{\lambda\gamma}{2\pi}C \tag{101.4}$$

and from the second equation that

$$D = \rho\frac{2\pi}{\lambda\gamma}(c - U)C \tag{101.41}$$

Substitution of these expressions in the first equation leads to the condition that

$$\gamma(c - U) = \left(\frac{2\pi}{\lambda}\right)^2\frac{c - U}{\gamma} \tag{101.42}$$

This condition is fulfilled in the first place if

$$c = U$$

i.e., if the wave travels with the speed of the undisturbed current, in other words, if the wave is at rest with respect to the moving fluid. This rather special case will not be considered further. The condition (101.42) is further satisfied if

$$\gamma = \pm\frac{2\pi}{\lambda} \tag{101.43}$$

Either of these alternatives gives a particular solution. Choosing first the positive sign and using (101.4) and (101.41),

$$
\begin{aligned}
u &= C\cos\frac{2\pi}{\lambda}(x - ct)e^{2\pi\frac{z}{\lambda}} \\
w &= C\sin\frac{2\pi}{\lambda}(x - ct)e^{2\pi\frac{z}{\lambda}} \\
p &= \rho(c - U)C\cos\frac{2\pi}{\lambda}(x - ct)e^{2\pi\frac{z}{\lambda}}
\end{aligned}
\tag{101.5}
$$

With the negative sign,

$$
\begin{aligned}
u &= -C' \cos \frac{2\pi}{\lambda}(x - ct)e^{-2\pi\frac{z}{\lambda}} \\
w &= C' \sin \frac{2\pi}{\lambda}(x - ct)e^{-2\pi\frac{z}{\lambda}} \\
p &= -\rho(c - U)C' \cos \frac{2\pi}{\lambda}(x - ct)e^{-2\pi\frac{z}{\lambda}}
\end{aligned}
\tag{101.51}
$$

The amplitude is denoted by C' in (101.51) in order to indicate that it is arbitrary and not necessarily equal to C.

Any linear combination of the two particular solutions (101.5) and (101.51) is a more general solution. Because C and C' are arbitrary in any case, it is not necessary to multiply them by arbitrary factors before adding in order to obtain the most general linear combination. [Strictly speaking, the expressions (101.5) and (101.51), C and C' being omitted, are particular solutions.]

Thus, the general solution of our problem is given by

$$
\begin{aligned}
u &= (Ce^{2\pi\frac{z}{\lambda}} - C'e^{-2\pi\frac{z}{\lambda}}) \cos \frac{2\pi}{\lambda}(x - ct) \\
w &= (Ce^{2\pi\frac{z}{\lambda}} + C'e^{-2\pi\frac{z}{\lambda}}) \sin \frac{2\pi}{\lambda}(x - ct) \\
p &= \rho(c - U)(Ce^{2\pi\frac{z}{\lambda}} - C'e^{-2\pi\frac{z}{\lambda}}) \cos \frac{2\pi}{\lambda}(x - ct)
\end{aligned}
\tag{101.6}
$$

These expressions have to satisfy the boundary conditions, at the rigid lower surface and at the free upper surface.

Without referring to the general boundary conditions of the preceding section, it is obvious that at the rigid lower boundary where $z = 0$ the velocity normal to the boundary must vanish. Because the boundary is horizontal, it follows that

$$w = 0 \qquad \text{for} \qquad z = 0$$

or

$$C = -C'$$

according to the second of Eqs. (101.6).

The undisturbed pressure at any level z is given by

$$P = g\rho(h - z)$$

as follows from (101.2) and from the condition that the pressure vanishes at the free surface.[1] The height of the free surface is h in the undisturbed state. In the disturbed state, let the height be z. The equation for the free surface becomes, according to (101.31) and (101.6),

$$P + p = g\rho(h - z) + \rho(c - U)C(e^{2\pi\frac{z}{\lambda}} + e^{-2\pi\frac{z}{\lambda}}) \cos \frac{2\pi}{\lambda}(x - ct)$$
$$= \text{const}$$

This expression can be simplified. Because p is a small quantity and the height z of the disturbed free surface is only slightly different from the height h of the undisturbed free surface, only an error of higher order is committed when z is replaced by h in the term representing p. This simplified equation for the free surface is to be substituted in the boundary condition (100.7) which in the present case reduces to

$$\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x} + w\frac{\partial P}{\partial z} = 0$$

In this equation the height of the disturbed boundary may be replaced by h in the expression for w. It follows, then, from the boundary condition that

$$\frac{2\pi}{\lambda}(c - U)^2\rho(e^{2\pi\frac{h}{\lambda}} + e^{-2\pi\frac{h}{\lambda}}) - g\rho(e^{2\pi\frac{h}{\lambda}} - e^{-2\pi\frac{h}{\lambda}}) = 0$$

or

$$(c - U)^2 = \frac{g\lambda}{2\pi} \tanh 2\pi\frac{h}{\lambda}$$

where the hyperbolic tangent

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Thus,

$$c = U \pm \sqrt{\frac{g\lambda}{2\pi} \tanh 2\pi\frac{h}{\lambda}} \qquad (101.7)$$

The wave velocity c consists of two terms. The first of these terms, U, states that the wave system is carried along by the

[1] The condition that the pressure at the free surface is equal to a constant different from zero would not alter the following calculations.

current. It is called the *convective* term. The second term represents the effect of gravity as shown by the factor g. It is called the *dynamic* term because it contains the force acting on the wave. These surface waves are also called "gravitational waves," for gravity is the force governing the wave motion.

If the fluid in the undisturbed state is at rest, $U = 0$, the wave may travel in the positive or negative x-direction as shown by the alternative sign before the dynamic term in (101.7). The wave velocity is the same in both cases. It depends on the wave length λ and on the depth of the fluid.

Two extreme cases may be considered separately. With increasing x, $\tanh x$ tends to unity. Thus, as the depth h of the fluid increases relative to the wave length λ, (101.7) becomes

$$c = U \pm \sqrt{\frac{g\lambda}{2\pi}} \tag{101.71}$$

This relation is known as the formula for the velocity of waves in deep water derived by Stokes. The expression "deep" water is perhaps somewhat misleading. For if $2\pi \frac{h}{\lambda} \geqq 2.5$, the formula (101.71) may be used with an error of about 1 per cent, for

$$\tanh 2.5 = 0.987$$

But, in this case,

$$h \geqq 0.4\lambda \tag{101.72}$$

Thus the expression (101.71) for the wave velocity in deep water may be used as soon as the depth of the water is more than four-tenths of the wave length.

For sufficiently small values of x,

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 + x - (1 - x)}{1 + x + 1 - x} = x$$

Equation (101.71) is then reduced to

$$c = U \pm \sqrt{gh} \tag{101.8}$$

provided that h is sufficiently small compared with λ. The formula (101.8) gives the velocity of "long waves" in shallow

water. It was derived by Lagrange. The above approximate expression for tanh x is correct to 1 per cent if $x \leqq 0.150$ or if

$$h \leqq 0.024\lambda \tag{101.81}$$

If the height of the homogeneous atmosphere (Prob. 3)

$$H = \frac{RT_0}{g}$$

where R = gas constant for air, T_0 = surface-air temperature is substituted for h in (101.8),

$$c = U \pm \sqrt{RT_0} \tag{101.9}$$

With a surface temperature of 20°C, $\sqrt{RT_0} = 290$ m/sec. The velocity of sufficiently long waves in the atmosphere is equal to the Newtonian velocity of sound $\sqrt{RT_0}$, at least in an incompressible, homogeneous atmosphere. V. Bjerknes[1] has shown that the velocity of long waves is equal to the Newtonian velocity of sound in any autobarotropic atmosphere. This more general validity of the result is easily understandable, for in any autobarotropic atmosphere as well as in an incompressible, homogeneous atmosphere a particle displaced from its original position is in equilibrium with its new surroundings.

102. Wave Motion at an Internal Surface of Discontinuity. Shearing Waves. We shall now consider the wave motion at an internal surface of discontinuity between two fluid layers. The density and the undisturbed velocity may be ρ and U in the lower layer and ρ' and U' in the upper layer. It follows that

$$\rho > \rho'$$

for the lower layer must be heavier than the upper layer to ensure a stable stratification. In the atmosphere, similar conditions are found at inversions where a warmer layer of air is situated above a colder one. The effect of the earth's rotation will again be neglected. This is permissible as long as the trajectories of the particles do not extend over too large an area.

The surface of discontinuity is horizontal as long as the motion is undisturbed. If the undisturbed boundary is chosen as the xy-plane, its equation can be written

$$z = 0$$

[1] Bjerknes, V., *Geofys. Pub.*, **3**, 3, 1923.

It will further be assumed that both layers are infinitely deep, so that the lower layer extends toward $z = -\infty$ and the upper one toward $z = +\infty$. Thus the effects of a lower rigid boundary and of an upper free surface are eliminated. The practical applications, however, are not seriously limited by this assumption if the waves are not too long; for, according to (101.72), the depth of a layer need be only four-tenths of the wave length in order to be considered as infinitely deep. A layer 1 km thick may still be considered as infinitely deep if the waves are 2.5 km long.

The perturbation equations of the problem are again of the form (101.3), and the general solution is therefore given by (101.6). The perturbation motion should not become infinite at an infinitely great distance from the surface of discontinuity. It follows that for the lower layer, where $z \leqq 0$, $C' = 0$. With this boundary condition at infinity the solution for the lower layer is reduced to

$$\begin{aligned} u &= Ce^{2\pi\frac{z}{\lambda}} \cos \frac{2\pi}{\lambda}(x - ct) \\ w &= Ce^{2\pi\frac{z}{\lambda}} \sin \frac{2\pi}{\lambda}(x - ct) \\ p &= \rho(c - U)Ce^{2\pi\frac{z}{\lambda}} \cos \frac{2\pi}{\lambda}(x - ct) \end{aligned} \tag{102.1}$$

Similarly, for the upper layer, $C = 0$. Because the quantities referring to the upper layer are denoted by dashes

$$\begin{aligned} u' &= -C'e^{-2\pi\frac{z}{\lambda}} \cos \frac{2\pi}{\lambda}(x - ct) \\ w' &= C'e^{-2\pi\frac{z}{\lambda}} \sin \frac{2\pi}{\lambda}(x - ct) \\ p' &= -\rho'(c - U')C'e^{-2\pi\frac{z}{\lambda}} \cos \frac{2\pi}{\lambda}(x - ct) \end{aligned} \tag{102.2}$$

Besides the two boundary conditions which state that the perturbation motion should remain finite at $z = +\infty$ and $-\infty$, there are two more conditions at the internal surface of discontinuity. These conditions may be used in the form (100.8). The equation of the internal boundary can be written, according to (100.2), in the form

$$P - P' + p - p' = -g(\rho - \rho')z + [\rho(c - U)C + \rho'(c - U')C'] \cos \frac{2\pi}{\lambda}(x - ct) = 0 \quad (102.3)$$

In this equation, the exponentials $e^{2\pi\frac{z}{\lambda}}$ and $e^{-2\pi\frac{z}{\lambda}}$ should appear in the expressions for p and p'. Here z denotes the height of the disturbed surface of discontinuity. But only an error of higher order is committed by putting $z = 0$, for the position of the disturbed surface of discontinuity deviates only slightly from the undisturbed position $z = 0$.

Upon substituting (102.3) into (100.8) and in the equation expressing the corresponding condition for the upper layer, it follows that

$$\frac{2\pi}{\lambda}(c - U)[\rho(c - U)C + \rho'(c - U')C'] - g(\rho - \rho')C = 0$$

$$(102.31)$$

$$\frac{2\pi}{\lambda}(c - U')[\rho(c - U)C + \rho'(c - U')C'] - g(\rho - \rho')C' = 0$$

These equations have been simplified by choosing for w and w' the value at the undisturbed internal-boundary surface instead of the value at the actual disturbed position which involves again only an error of higher order. Upon equating the ratios of C/C' following from these two equations the relation between c and λ becomes, after simplification,

$$(c - U')^2\rho' + (c - U)^2\rho - \frac{g\lambda}{2\pi}(\rho - \rho') = 0$$

or, upon rearranging,

$$c^2(\rho + \rho') - 2c(\rho U + \rho' U') + U^2\rho + U'^2\rho' - \frac{g\lambda}{2\pi}(\rho - \rho') = 0 \quad (102.4)$$

Thus,

$$c = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g\lambda}{2\pi}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}} \quad (102.5)$$

The expression for the wave velocity consists again, as in the preceding section, of a convective and a dynamic term. The convective term represents the mean value of the undisturbed velocity in

both layers. The dynamic term is due to the effect of the gravitational force of the earth and due to the wind difference on both sides of the surface of discontinuity. Because such a wind difference is called a "shear," its effect on the wave motion may be called the "shearing effect."

If $\rho' = 0$, (102.5) becomes identical with (101.71), the formula for the wave velocity on the free surface of one layer whose depth is infinite.

To discuss the implications of (102.5), some special cases may be considered. If

$$U = U' = 0,$$

$$c = \pm\sqrt{\frac{g\lambda}{2\pi}\frac{\rho - \rho'}{\rho + \rho'}} \tag{102.51}$$

This formula differs from the corresponding formula (101.71) for waves at the free surface of an infinitely deep fluid by the factor $\sqrt{\dfrac{\rho - \rho'}{\rho + \rho'}}$. The waves are purely gravitational in both cases. Since $\rho = P/RT$ and $\rho' = P'/RT'$ where $P = P'$, (102.51) may be written

$$c = \pm\sqrt{\frac{g\lambda}{2\pi}\frac{T' - T}{T' + T}} \tag{102.52}$$

If $T' = 283°$ abs above the surface of discontinuity and $T = 273°$ abs below, $\sqrt{\dfrac{T' - T}{T' + T}} = 0.13$. Thus, for the same wave length, internal waves are considerably slower than waves at the free surface.

When $\rho' > \rho$, *i.e.*, when the upper fluid is heavier than the lower one, c becomes imaginary, and the argument of the periodic function in (102.1) and (102.2) is complex. The perturbation quantities change then from periodic to exponential functions of the time, containing terms multiplied by $e^{\frac{2\pi}{\lambda}ct}$ and by $e^{-\frac{2\pi}{\lambda}ct}$. The latter terms decrease with time, and the former increase. Because the perturbation increases with time, the waves are unstable. This result is, of course, to be expected in view of the gravitational instability of the stratification.

The directions of the horizontal perturbation velocities u' and u in the upper and the lower layer are opposite. From the

first equation of (102.31), it follows that

$$\frac{C'}{C} = -\frac{c^2\rho - (g\lambda/2\pi)(\rho - \rho')}{\rho' c^2}$$

or, with (102.51),

$$\frac{C'}{C} = 1$$

Therefore, Eqs. (102.1) and (102.2) show that the horizontal components of the perturbation motion have opposite signs in both layers. The result holds also if U and U' are different from zero.

If the density in both layers is the same but the wind velocity is different,

$$c = \frac{U + U'}{2} \pm i\,\frac{U - U'}{2} \tag{102.6}$$

Because the dynamic term of the wave velocity depends now on the wind shear, these waves are called "shearing waves." They are also unstable, for the dynamic term is imaginary.

The waves whose velocity is represented by the more general formula (102.5) are of a mixed shearing and gravitational type. They are stable if the expression under the square root is positive, *i.e.*, when

$$\lambda \geqq \frac{2\pi}{g}\,\frac{\rho\rho'(U - U')^2}{(\rho - \rho')(\rho + \rho')} \tag{102.61}$$

Thus, sufficiently small waves are unstable, owing to the effect of shearing. If the temperature is introduced in (102.61) instead of the density,

$$\lambda \geqq \frac{2\pi}{g}\,\frac{(U - U')^2 TT'}{(T' - T)(T' + T)} \tag{102.62}$$

The following table gives the limiting wave length for instability when $T = 273°$ abs. Shorter waves are unstable, and longer ones stable.

For longer waves the effect of the gravitational stability on the stratification, which is represented by the first term under the square root of (102.5), becomes more effective, and the waves become stable when (102.61) is satisfied.

LIMITING WAVE LENGTH, m

ΔT, deg C	ΔU, m/sec				
	4	8	12	16	20
4	352	1408	3168	5616	8789
8	177	709	1595	2829	4426
12	119	475	1073	1901	2974
16	90	359	808	1431	2244
20	73	289	652	1156	1809

103. Billow Clouds. If the velocity of the waves $c = 0$, the wave length can be determined from the observable wind velocities and densities. Because the time factor in the expressions for the perturbation quantities then vanishes, the motion is steady. The length of the waves under these conditions is most easily found from (102.4). If $c = 0$, it follows that

$$\lambda = \frac{2\pi}{g} \frac{U^2\rho + U'^2\rho'}{\rho - \rho'} \tag{103.1}$$

or if the temperature is introduced instead of the density,

$$\lambda = \frac{2\pi}{g} \frac{U^2T' + U'^2T}{T' - T} \tag{103.2}$$

Atmospheric waves to which this formula may be applied are mainly observed in the form of billow clouds which appear at the boundary of an inversion. It may be assumed that the cloud system moves with the mean velocity $\dfrac{U + U'}{2}$ of the upper and lower layer so that the wave velocity vanishes in a coordinate system moving with the velocity $\dfrac{U + U'}{2}$ of the cloud system. Condensation and cloud formation take place where the air ascends, while the sky is clear where the wave motion causes descent of air. When the temperatures and wind velocities are observed above and below the inversion, the wave length can be computed and compared with the observed wave length. Wegener[1] first made such a comparison. The observed wave

[1] WEGENER, A., *Beitr. Phys. Atm.*, **2**, 55, 1906–1908.

lengths are, in general, smaller than the computed ones, the difference being larger the smaller the amount of the inversion. Haurwitz[1] has shown that this is due to the assumption that the atmosphere is incompressible and that the only variation in density occurs at the inversion. In making this assumption, the stability of the atmosphere is taken into consideration only in so far as it is due to the density discontinuity at the inversion; and, consequently, the computed wave lengths are too large. In reality the stability of the atmosphere is in part due to the fact that the actual lapse rate is smaller than the adiabatic. This stabilizing effect can be taken into account if the atmosphere is treated as a compressible fluid. The computed wave lengths are then in much better agreement with the observed ones, as shown in the following table. The first column of this table gives the date of the observation, the second the temperature discontinuity, the third the wind discontinuity, the fourth the observed wave length, and the fifth and sixth the computed wave lengths under the assumption of an incompressible and compressible fluid.

Wave Lengths of Billow Clouds*

Date	ΔT, deg C	ΔU, m/sec	$\lambda_{\text{observed}}$, m	λ_{incomp}, m	λ_{comp}, m
Feb. 17, 1894	2.6	14.8	1000–2000	6800	2190
Dec. 6, 1905	2.2	10.4	1600	4210	1570
Feb. 12, 1906	1.0	4.0	710	1380	601
Feb. 19, 1906	3.7	3.0	175	212	196
Feb. 3, 1910	4.6	6.7	1490	1690	1360
Jan. 17, 1918	4.2	5.5	570	620	545
Jan. 18, 1918	0.4	6.0	2060	7300	1510
Jan. 27, 1918	0.1	6.0	940	31500	1000
May 7, 1931	0.0	6.0	2000	∞	1030

* Haurwitz, B., *Met. Z.*, **48**, 483, 1931.

Such billows may sometimes be responsible for periodic ceiling fluctuations, as pointed out by Jacobs.[2]

104. An Example of Inertia Waves. We shall next consider the effect of the earth's rotation. To simplify matters the

[1] Haurwitz, B., *Gerl. Beitr. Geophys.*, **34**, 213, 1931; **37**, 16, 1932.

[2] Jacobs, W. C., *Monthly Weather Rev.*, **65**, 9, 1937.

motion at the pole will be studied. Here, only the vertical component $\Omega_z = \omega$ of the earth's rotation does not vanish, and therefore no complications due to the horizontal components arise. The fluid is again assumed to be incompressible and homogeneous. It may be enclosed between two rigid horizontal boundaries

$$z = 0 \quad \text{and} \quad z = h$$

Under these conditions, oscillations are possible in a homogeneous, incompressible fluid only if the whole fluid system rotates, as will be seen below.

In the undisturbed state the fluid may be at rest; thus, according to (99.3),

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g$$

In the disturbed state the variables may again be regarded as independent of the y-coordinate. The perturbation equations become, with the aid of (99.5),

$$\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + 2\omega u &= 0 \\ \frac{\partial w}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \end{aligned} \tag{104.1}$$

In these equations the y-component of the velocity v has been included, although it has been assumed that it is independent of y and only a function of x, z, and t. If v were assumed equal to zero, u would also vanish according to the second equation; w could be only a function of x according to the equation of continuity. However, since the fluid has rigid horizontal boundaries, it follows that

$$w = 0 \quad \text{if} \quad z = 0 \quad \text{and} \quad z = h \tag{104.2}$$

as is obvious without reference to the more general boundary conditions of Sec. 100. Thus, w would have to vanish too if it were only a function of x. Therefore, it must be assumed that $v \neq 0$, if the effect of the earth's rotation is taken into account. Owing to the earth's rotation the motion no longer takes place

in a vertical plane, as in the examples considered in the two preceding sections.

The boundary conditions (104.2) show that w cannot be a true exponential function of z as in the preceding sections. Actually, if a dependence of the form $e^{\gamma z}$ were to be assumed, the subsequent calculation would show that γ must be imaginary in order to satisfy (104.2). The dependence on z must be given by a trigonometric function. If we assume that

$$w \sim \sin\left(n\pi \frac{z}{h}\right) \qquad \text{when} \qquad n = 1, 2, 3, \cdots$$

the boundary conditions are satisfied. From (104.1), it is seen that u, v, and p must then be proportional to $\cos\left(n\pi \frac{z}{h}\right)$. Therefore, let

$$\begin{aligned} u &= A \cos \frac{2\pi}{\lambda}(x - ct) \cos\left(n\pi \frac{z}{h}\right) \\ v &= B \sin \frac{2\pi}{\lambda}(x - ct) \cos\left(n\pi \frac{z}{h}\right) \\ w &= C \sin \frac{2\pi}{\lambda}(x - ct) \sin\left(n\pi \frac{z}{h}\right) \\ p &= D \cos \frac{2\pi}{\lambda}(x - ct) \cos\left(n\pi \frac{z}{h}\right) \end{aligned} \tag{104.3}$$

where A, B, C, and D are constants as in the preceding section. When $n = 1$, w vanishes at the lower and upper boundaries. For larger n, it vanishes also at h/n, $2h/n$, $\cdots$, $(n - 1)h/n$. These horizontal surfaces represent $n - 1$ nodal planes of the vertical motion; u, v, and p have maxima or minima on these planes.

Upon substituting (104.3) in (104.1), it follows that

$$\begin{aligned} \frac{2\pi}{\lambda} cA - 2\omega B &= \frac{1}{\rho}\frac{2\pi}{\lambda} D \\ -\frac{2\pi}{\lambda} cB + 2\omega A &= 0 \\ -\frac{2\pi}{\lambda} cC &= \frac{1}{\rho}\frac{n\pi}{h} D \\ -\frac{2\pi}{\lambda} A + \frac{n\pi}{h} C &= 0 \end{aligned}$$

If A, B, and D are expressed by C, according to the fourth of these equations,

$$A = \frac{\lambda n}{2h} C \tag{104.31}$$

according to the third equation,

$$D = -\rho \frac{2hc}{n\lambda} C \tag{104.32}$$

and according to the second and fourth equations,

$$B = \omega \frac{\lambda^2 n}{\pi c 2h} C \tag{104.33}$$

Upon substituting these values for A, B, and D in the first equation, a relation for the determination of c is obtained,

$$c \frac{\pi n}{h} - \omega^2 \frac{\lambda^2 n}{\pi c h} = -\frac{4\pi c h}{\lambda^2 n}$$

Thus,

$$c = \frac{\omega\lambda}{\pi\sqrt{1 + (4h^2/n^2\lambda^2)}} \tag{104.4}$$

When $\omega = 0$, *i.e.*, when the effect of the earth's rotation is disregarded, $c = 0$ and no oscillation is possible, for u, v, and w are then independent of the time. Furthermore, the amplitude D of the perturbation pressure would vanish since it is proportional to c. Only a stationary wave pattern of the streamlines could exist if $\omega = 0$, but no oscillation. The rotation is essential for this type of wave motion.

The greater the height of the layer, the smaller the velocity of these waves for one and the same wave length. When nodal planes of the vertical motion exist, $n > 1$, the wave velocity is the same as in a layer corresponding to the distance between two nodal planes. This is to be expected, for each nodal plane can be replaced by a rigid boundary. Upon choosing $h = 8$ km, approximately the height of the homogeneous atmosphere under average conditions, the following velocities are found for various wave lengths:

λ, km	1	10	100	500	1000
c, m/sec	1.45×10^{-3}	1.23×10^{-1}	2.29	11.6	23.2

Because the period of the oscillation

$$\tau = \frac{\lambda}{c}$$

it follows that

$$\tau = \frac{\pi}{\omega}\sqrt{1 + \frac{4h^2}{n^2\lambda^2}} \tag{104.41}$$

Because $2\pi/\omega$ is the length of the sidereal day at the pole, Eq. (104.41) shows that the period of the oscillations considered in this section cannot be smaller than half a sidereal day. At latitudes different from that of the pole, ω is multiplied by the sine of the latitude, $\sin\varphi$. Because $2\pi/\omega \sin\varphi$ is the length of the pendulum day, the period at any latitude must be larger than half the pendulum day.

105. General Discussion of Inertia Waves. The type of wave motion considered in Sec. 104 is due to the Coriolis force of the earth's rotation and could not exist in its absence. Because the Coriolis force is due to inertia of the mass, the name "inertia waves" has been given to oscillations of this type by V. Bjerknes[1] and Solberg.[2] Inertia plays here a role similar to that of gravitation in reference to the wave types considered in the preceding sections.

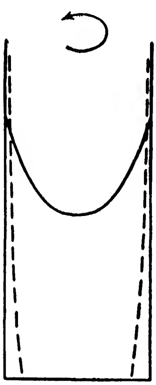

FIG. 71.—Effect of rotation on the surface of a fluid. Full curve, slow rotation; broken curve, fast rotation.

In view of the importance of the inertia waves for the understanding of the wave theory of cyclones, this wave type may be considered from another angle.

If a hollow cylinder of circular cross section is partly filled with a fluid and rotated around its axis, the fluid will be set in rotation, too, owing to friction at the walls and at the bottom of the cylinder. The free surface forms a parabolic surface, this being the equilibrium surface of a fluid subjected to the influence of gravity and centrifugal force. When the rotation is sufficiently fast, the fluid is pressed very strongly against the walls of the cylinder and its surface assumes a practically cylindri-

[1] BJERKNES, V., and collaborators, *op. cit.*, p. 422.

[2] SOLBERG, H., and BJERKNES, V., *Avh. Norske Vid. Akad., Math.-Nat. Kl.*, vol. I, No. 7, 1929.

cal shape, as shown schematically in Fig. 71. The gravitational force is so much smaller than the centrifugal force that its effect becomes negligible if the rotation is sufficiently fast. Thus, instead of the horizontal surfaces of fluids which occur when the influence of gravity is preponderant, the fluid in the rapidly rotating cylinder has a practically vertical surface due to the action of the centrifugal force in the horizontal direction. If such a vertical fluid surface is subjected to a small disturbance, a wave motion will originate. This process is quite analogous to the formation of waves on a horizontal surface; in this case, however, the energy of the wave motion is gravitational, whereas in the case of the rotating cylinder the centrifugal force replaces the gravitational force.

The simple inertia waves between two rigid horizontal boundaries on the rotating earth which were derived in the preceding section are stable, for c cannot become complex, according to (104.4). However, a more general discussion of the stability or instability of the inertia waves is necessary in view of their importance for the cyclone problem.[1]

Consider a fluid mass rotating around a vertical axis with the angular velocity η which is only a function of the distance R from the axis of rotation. If the fluid is enclosed between rigid horizontal boundaries and if $\eta = \omega =$ const everywhere, we have just the case considered in the preceding section; but this simplifying assumption will be abandoned now.

The angular momentum of a mass is constant if no forces are acting on it (Sec. 44).

$$\eta R^2 = \text{const} \tag{105.1}$$

where R is the distance from the axis of rotation. If the particle is displaced to a distance $R + r$ from the axis while retaining its angular momentum, its angular velocity becomes η', and

$$(R + r)^2\eta' = R^2\eta$$

$$\eta' = \frac{R^2}{(R + r)^2}\,\eta \tag{105.2}$$

The centrifugal force acting on the displaced particle is

$$\eta'^2(R + r) = \eta^2\,\frac{R^4}{(R + r)^3} \tag{105.3}$$

[1] See also V. Bjerknes and collaborators, *op. cit.*, pp. 163–167.

The centrifugal force on the surrounding fluid mass at the distance $R + r$ is

$$\eta^{*2}(R + r)$$

where η^* is the angular velocity of the fluid at the distance $R + r$ from the axis. Stability prevails when the centrifugal force on the surrounding fluid is larger than that on the displaced mass,

$$\eta^{*2}(R + r) > \eta^2 \frac{R^4}{(R + r)^3} \tag{105.4}$$

For if (105.4) is satisfied, the displaced particle will be pushed back to its original position because the centrifugal force acting on it is smaller than the centrifugal force on the surrounding air. While being forced back, the particle acquires kinetic energy and therefore will overshoot its equilibrium position. In this manner a stable oscillation to and fro about the equilibrium position is brought about. This type of stability is called *dynamic stability.*

If the centrifugal force acting on the particle were larger than the centrifugal force acting on the environment, the particle would be moved farther from its original position. The distribution of the angular velocity would tend to create instability. If finally the centrifugal force on the particle were equal to that on the surrounding fluid, no force would be acting on a displaced fluid particle; it would everywhere be in indifferent equilibrium.

The stability condition (105.4) can also be written in the form

$$\eta^*(R + r)^2 > \eta R^2 \tag{105.5}$$

which states that stability with respect to small perturbations exists if the angular momentum increases outward. This condition is fulfilled, for instance, when the angular velocity is constant. If the angular momentum is constant, we have indifferent equilibrium; if the angular momentum decreases outward, instability prevails.

The stability condition in the form (105.5) may be applied to the earth's atmosphere in order to decide if the effect of inertia on atmospheric wave motion is stabilizing or not. The distance from the axis

$$R = E \cos \varphi$$

where E is the earth's radius and φ the latitude. The total angular velocity

$$\eta = \dot{\lambda} + \omega$$

where $\dot{\lambda}$ is the angular velocity relative to the earth, and ω the angular velocity of the earth's rotation. The stability condition (105.5) becomes

$$(\dot{\lambda}^* + \omega) \cos^2 \varphi^* > (\dot{\lambda} + \omega) \cos^2 \varphi \qquad \text{when} \qquad \varphi^* < \varphi \qquad (105.6)$$

As long as the angular velocity $\dot{\lambda}$ relative to the earth vanishes, the condition is certainly satisfied. But even if $\dot{\lambda}$ is different from zero, it is always very much smaller than ω, except near the poles. At 60° latitude, for instance, the velocity due to the earth's rotation is about 232 m/sec, and the wind velocity is one-tenth or one-twentieth of this amount. In general, therefore inertia waves in the atmosphere are stable, and the effect of the Coriolis force on atmospheric wave motions is stabilizing.

106. Large-scale Oscillations of the Atmosphere. In Sec. 58 (page 163), it was pointed out that, in large atmospheric disturbances, convergence and divergence must arise even if the wind field is geostrophic, owing to the increase of the Coriolis force with the latitude. It was shown in a qualitative fashion that, owing to this effect, a pressure field with west-east isobars on which sinusoidal disturbances are superimposed would move toward the west.

Instead of using the perturbation equations (99.5) and (99.6) directly to solve this problem, we may apply Eq. (85.2) here,

$$\frac{d}{dt}(\zeta + 2\omega \sin \varphi) + \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)(\zeta + 2\omega \sin \varphi) = 0 \qquad (85.2)$$

where $\zeta = \dfrac{\partial \bar{v}}{\partial x} - \dfrac{\partial \bar{u}}{\partial y}$. Bars are added to the velocity components in order to indicate that they refer to the total disturbed plus undisturbed motion. The fluid will be considered incompressible as heretofore. The x-axis of the coordinate system points toward the east, the y-axis toward the north, and the z-axis vertically upward. We shall further assume that the total motion consists of an undisturbed zonal current U of constant

velocity[1] and a horizontal perturbation motion with the components u and v

$$\bar{u} = U + u, \qquad \bar{v} = v, \qquad \bar{w} = 0 \tag{106.1}$$

The components of the perturbation velocity may be so small that terms containing them in the second or higher order can be neglected. Because $\bar{w} = 0$,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{106.11}$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}\right)(\zeta + 2\omega \sin \varphi) = 0 \tag{106.2}$$

The Coriolis parameter $\omega \sin \varphi$ depends only on y owing to the orientation of the coordinate system

$$\frac{\partial}{\partial y}(2\omega \sin \varphi) = \frac{\partial(2\omega \sin \varphi)}{E\, \partial\varphi} = \frac{2\omega \cos \varphi}{E} = \sigma \tag{106.3}$$

σ varies also with the latitude; but, in the following calculation, this variation will be neglected. From (106.2), it follows that, if terms of second and higher order in the perturbation quantities are omitted,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\sigma = 0 \tag{106.4}$$

The equation of continuity for the perturbation motion,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

can be satisfied by putting

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x} \tag{106.5}$$

where ψ is a stream function. Then (106.4) becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \sigma\frac{\partial \psi}{\partial x} = 0 \tag{106.51}$$

[1] ROSSBY, C.-G., *J. Mar. Research*, **2**, 38, 1939. See also B. Haurwitz, *J. Mar. Research*, **3**, 35, 1940.

A solution of this equation can be assumed in the form

$$\psi = C \cos \frac{2\pi}{\lambda}(x - ct) \cos \frac{2\pi}{\delta} y \tag{106.6}$$

λ is the wave length, and δ may be called the width of the disturbance although it represents actually the width of two adjoining centers in which the motion is in opposite directions. Upon substituting (106.6) in (106.51), it follows that

$$c = U - \frac{\sigma}{4\pi^2}\lambda^2 \frac{\delta^2}{\lambda^2 + \delta^2} \tag{106.61}$$

The velocity c of the disturbance is composed of the geostrophic wind current U and a dynamic term that depends on the latitudinal variation σ of the Coriolis parameter. This dynamic term increases with the wave length and toward the equator. It decreases as the ratio λ/δ increases.

The total number of waves n around the circumference of the earth at the latitude is given by

$$n\lambda = 2\pi E \cos \varphi$$

Substitution in (106.61) shows that

$$c = U - \frac{2\omega E}{n^2} \cos^3 \varphi \frac{1}{1 + \lambda^2/\delta^2} \tag{106.62}$$

If the lateral extent of the disturbance is infinite $\delta = \infty$, *i.e.*, if the perturbation does not depend on y, the following values of $U - c$ are obtained at different latitudes and for different wave numbers:

WAVE VELOCITIES FOR DIFFERENT WAVE NUMBERS AND AT DIFFERENT LATITUDES. AFTER ROSSBY

φ, deg	n = 2	3	4	5	6	7
	Velocity, m/sec					
30	150.7	67.0	37.7	24.1	16.7	12.8
45	82.0	36.5	20.5	13.1	9.1	6.7
60	29.0	12.9	7.3	4.6	3.2	2.4

If the lateral extent of the disturbance is not infinite, the figures of the preceding table are reduced by the factor $\frac{1}{1 + \lambda^2/\delta^2}$.

Equation (106.61) shows that the disturbances become stationary when

$$\lambda = 2\pi \sqrt{\frac{U}{\sigma}} \sqrt{1 + \frac{\lambda^2}{\delta^2}} \tag{106.7}$$

Some stationary wave lengths at different gradient-wind velocities and different latitudes when $\delta = \infty$ are shown in the following table.

STATIONARY WAVE LENGTH

φ, deg	U, m/sec				
	5	10	15	20	25
	Wave length, km				
30	3150	4460	5460	6310	7050
45	3490	4940	6050	6980	7790
60	4140	5850	7180	8300	9260

When the disturbance is of finite width, the stationary wave lengths are larger. If, for instance, the width is equal to the wave length, the stationary wave lengths are 40 per cent larger.[1] The preceding formulas explain to a certain extent the behavior of the so-called "centers of action," like the Icelandic and the Aleutian lows, and the Pacific and Azores highs. If (106.7) is satisfied, the center remains stationary. But if the eastward drift U of the general circulation becomes stronger than necessary, according to (106.7), the center will be displaced eastward; if U becomes smaller, the center will be displaced westward. The zonal velocity U of the general circulation or the meridional pressure gradient to which U is closely related by the geostrophic wind relation is therefore an important index of the motion of the semipermanent centers of action that dominate the large-scale developments of the weather.[2]

[1] HAURWITZ, B., *Am. Geophys. Un. Trans.*, (2) 263, 1940.

[2] ALLEN, R. A., *Quart. J. Roy. Met. Soc.*, **66**, Suppl., 1940.

The disturbances determined by the preceding formulas may arise spontaneously, *i.e.*, without the action of a generating force. They are free disturbances, analogous to the free oscillations of a vibrating system. The heterogeneities of the earth's surface, in particular, the unequal distribution of water and land, tend to set up perturbations whose location is determined by these geographical factors. These heterogeneities may be treated as external forces that give rise to forced disturbances, analogous to the case of the forced oscillations of a vibrating system. Particularly strong disturbances will be caused by those forces whose dimensions are close to the dimensions of a free disturbance. The dimensions take over the role that the period has in the case of resonance of vibrating systems, for the forces due to the surface heterogeneities of the earth are independent of the time.

The previous formulas are developed for a flat earth. Obviously, when disturbances of such dimensions as those in the table for stationary wave lengths (page 298) are considered, this assumption is hardly justified except as a first approximation. The simple derivation given here can also be extended to a spherical earth, and it is possible to take into account at the same time the variation of σ with the latitude.[1] The results are the same qualitatively, but the quantitative results are different.

Problems

30. Find the wave velocity at the internal boundary between two homogeneous incompressible fluid layers enclosed between rigid lower and upper boundaries when the depth of each layer is small compared with the wave length and when the fluid is at rest in the undisturbed state. The effect of the earth's rotation should be neglected.

31. Find the wave velocity in an incompressible homogeneous fluid layer with a rigid lower boundary and a free upper surface when the undisturbed current increases linearly with the elevation. The effect of the earth's rotation should be neglected. In particular, what are the expressions for the wave velocity in a very shallow and in a very deep fluid layer?

[1] HAURWITZ, B., *J. Mar. Research*, **3**, 254, 1940.

CHAPTER XV

AIR MASSES, FRONTS, CYCLONES, AND ANTICYCLONES

107. Air Masses. A study of the weather maps shows that the properties of the air are often nearly the same over areas extending many thousands of miles. Such a uniform body of air is called an "air mass." The homogeneity is due to the life history of the air mass. When air stagnates for a considerable interval of time in a region where the earth's surface is homogeneous, it assumes uniformly the properties of this part of the earth's surface. The principal regions in which formation of air masses takes place are the tropical and subtropical zones on the one side, the arctic and subarctic regions on the other side. Consequently, the two principal types of air mass are the tropical and the polar air masses. In middle latitudes the uniformity of conditions and the light winds are generally lacking so that no or only less important source regions are found here.

The tropical and polar air masses may be further subdivided, according to whether the source region is a land or a water surface. In the first case the air is called "continental," in the second case "maritime." Continental air masses are, of course, drier than maritime air masses, other things being equal.

For a complete description of air masses, their origin and their properties, the reader is referred to the textbooks on weather forecasting.[1] The very brief remarks made here are intended only as an introduction to the discussion following.

The two main processes active in the formation of air masses are turbulent mass exchange and radiation. Turbulent mass exchange transports heat from the ground and the lowest layers upward (Sec. 84). Similarly, the effects of the cooling process are extended upward. But in this case the transfer will not be so strong because cooling from below increases the vertical

[1] For instance, S. Petterssen, "Weather Analysis and Forecasting," McGraw-Hill Book Company, Inc., New York, 1940; J. Namias, "Air Mass and Isentropic Analysis," 5th ed., American Meteorological Society, Milton, Mass., 1940.

stability of the air. Thus, the turbulent heat transfer is reduced, especially if an inversion is formed (page 231). Besides heat, the water vapor evaporated from the surface into the lowest layers of the ground is also carried upward by turbulent mass exchange.

The role of radiation in the development of air masses becomes particularly important when the effect of the turbulent heat transport is very small, as in polar air that is cooled at the ground (Sec. 39).

Once the air mass leaves its source region, it undergoes a continuous transformation. In general, tropical air masses moving toward higher latitudes come gradually in contact with cooler parts of the earth's surface. Thus, their temperature is lowered, first at the ground, so that the vertical stability is increased. Polar air masses advancing over warmer ground are heated from below so that their actual stability decreases (Sec. 84). Considerations of the life history of the air masses lead to the "differential" classification of air masses; the classification according to source regions is called the "geographical" classification.

Simultaneously with the heating or cooling of the advancing air mass, there is also, as a rule, a change of its moisture content. Generally speaking, a tropical air mass moving over a cooler surface will lose water vapor by condensation owing to cooling, whereas polar air will gain water vapor by evaporation. After these variations have gone on for some time, the air-mass properties are changed to such an extent that the air mass is given a special designation showing its transitional state.

A modification of temperature and humidity of air masses is, of course, brought about by vertical motions, also. If these motions take place adiabatically, the temperature increases with descent and decreases with ascent. Similarly, variations of the density of the water vapor, the absolute humidity, occur. In order to follow the air masses from day to day it is necessary to use "conservative" properties, as, for instance, equivalent potential temperature and wet-bulb potential temperature (see Secs. 24 and 25) which do not vary appreciably during adiabatic changes. Of course, nonadiabatic effects such as turbulent mass exchange or radiation also modify these so-called "conservative" properties. This fact has to be taken into account in analyzing weather maps.

108. Fronts and Their Origin. In one and the same air mass, there is, according to the meaning of the term "air mass," very little or no change of the properties of the air in the horizontal direction. But in passing from one air mass to another a change of the properties must necessarily occur, and the rapidity of the change depends on the width of the zone of transition. When the air masses are in their respective source regions, the variation of the properties from air mass to air mass will be closely equal to the regular average gradient determined by the mean distribution of the meteorological elements over the globe. When one or both of the air masses move from their source regions toward each other, the zone of transition becomes smaller, and the gradient of the properties larger. When the zone of transition becomes sufficiently small so that it appears as a line on the weather map, it is called a "front" and may be treated as a sharp line. An accurate criterion of when a zone of transition should be called a front cannot be given;[1] such a criterion would obviously depend on the density of the network of observing stations. However, it may be stated that transitional zones whose width is smaller than 50 mi may be regarded as fronts. Wider zones are frequently called "frontal" zones. It is necessary, of course, that across the front or frontal zone a significant change of the property take place.

Strictly speaking, a front is the intersection between the frontal surface—which separates two air masses of different properties—and the ground, although the term "front" is often used to designate the whole surface of separation, also. The dynamic and kinematic principles determining the position and the motion of frontal surfaces have been considered in Chaps. VIII and IX.

In order to form a frontal surface, two air masses of different properties must come sufficiently close together for a sharp and narrow zone of transition to come into existence. The kinematic conditions underlying the formation of atmospheric fronts have been studied by Bergeron,[2] Bjerknes,[3] and Petterssen.[4] Pet-

[1] See, for a detailed discussion, T. Bergeron, *Geofys. Pub.*, V, No. 6, 1928.

[2] BERGERON, *loc. cit.*

[3] BJERKNES, V., and collaborators, "Physikalische Hydrodynamik," §9, §176, Verlag Julius Springer, Berlin, 1933.

[4] PETTERSSEN, S., *Geofys. Pub.*, **11**, No. 6, 1935; "Weather Analysis and Forecasting," Chap. V.

terssen considers a property α which for the sake of simplicity may be regarded as conservative. The distribution of α in the horizontal is given by a function $\alpha\ (x,\ y)$ and can be represented on the map by a family of curves given by the condition that $\alpha =$ const. These curves will, in general, move across the map. If they move in such a manner that they tend to produce a discontinuity along a line on the chart, a front may be formed and we have "frontogenesis."

The frontogenetical effect may be measured by the variation of $\partial\alpha/\partial n$ with time, n being the direction at right angles to the curves α. The line of frontogenesis must always consist of the same particles; otherwise, the maximum frontogenetical effect would always act on new air, and no front would form. Thus, the variation with time of $\partial\alpha/\partial n$ must be the individual variation, and the frontogenetical effect may be expressed by

$$F = \frac{d}{dt}\left(\frac{\partial\alpha}{\partial n}\right) \tag{108.1}$$

In order to have frontogenesis, F must be positive, and the conditions for F to have a maximum along the line of frontogenesis must be satisfied. If F is negative in a given area, any existing fronts are weakened and eventually are dissolved and we have "frontolysis."

If v_s and v_n are the velocity components parallel and normal to the isopleths of α, (108.1) may be written explicitly

$$F = \left(\frac{\partial}{\partial t} + v_s\frac{\partial}{\partial s} + v_n\frac{\partial}{\partial n}\right)\frac{\partial\alpha}{\partial n} \tag{108.11}$$

The vertical velocity component can be disregarded, at least as long as frontogenesis at the ground is considered. Because α is a conservative property, its individual variation vanishes,

$$\frac{d\alpha}{dt} = \frac{\partial\alpha}{\partial t} + v_s\frac{\partial\alpha}{\partial s} + v_n\frac{\partial\alpha}{\partial n} = 0 \tag{108.2}$$

Upon carrying out the differentiation in (108.11) and eliminating $\partial\alpha/\partial t$ with the aid of (108.2), it follows that

$$F = -\frac{\partial v_n}{\partial n}\frac{\partial\alpha}{\partial n} \tag{108.3}$$

Since $\partial\alpha/\partial n > 0$, frontogenesis occurs when v_n decreases in the direction of increasing values of α. If v_n increases in the direction of increasing values of α, frontolysis takes place. These qualitative rules are easily derived directly by means of simple sketches.

It may now be assumed that the velocity field is linear; thus, the velocity components can be written in the form

$$u = u_0 + u_1x + u_2y$$
$$v = v_0 + v_1x + v_2y$$

The choice of a linear velocity field is, of course, a serious restriction of the generality of the subsequent results. But the above expressions for the velocity components may also be regarded as the first terms of a Taylor series for u and v, and therefore the

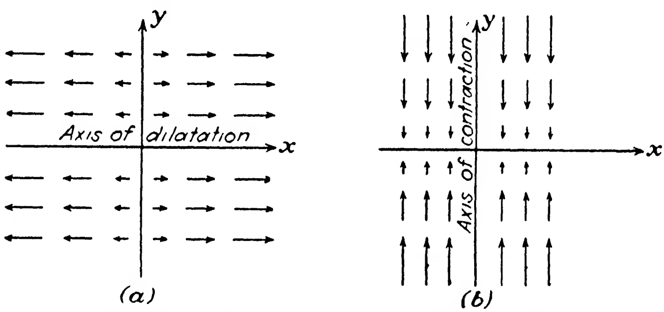

Fig. 72.—Deformation in a linear field of motion.

results may be applied to the neighborhood of a point for which the development of u and v into a Taylor series holds.

Furthermore, linear fields of motion seem to predominate in regions where the distribution of the isobars shows a saddle point, and in such areas fronts frequently form.

By a suitable choice of a coordinate system the above expressions for the velocity components may be written

$$\begin{aligned} u &= u_0 + ax + bx - cy \\ v &= v_0 - ay + by + cx \end{aligned} \qquad (108.4)$$

The terms u_0 and v_0 represent a translatory motion, and the terms ax and $-ay$ a deformation,

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2a$$

The x-component of the deformation velocity is positive in the positive x-direction and negative in the negative x-direction, and

therefore the x-axis is the axis of dilatation (Fig. 72a). Similarly the y-axis is the axis of contraction (Fig. 72b). The terms bx and by measure the divergence of the field (Sec. 50)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2b$$

and $-cy$ and cx its vorticity (Sec. 51)

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2c$$

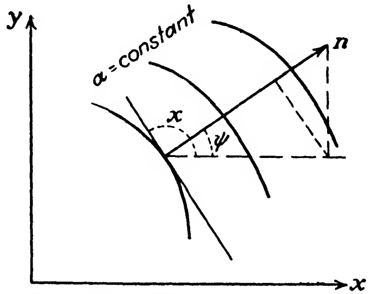

FIG. 73.—Transformation of the expression for the frontogenetical effect.

In order to substitute the expression for the linear field of motion (108.4) in the expression for the frontogenetical effect (108.3), v_n has to be transformed. Let ψ be the angle between the axis of dilatation and n (Fig. 73). Then

$$v_n = u \cos \psi + v \sin \psi$$

and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial}{\partial y}\frac{\partial y}{\partial n} = \cos \psi \frac{\partial}{\partial x} + \sin \psi \frac{\partial}{\partial y}$$

Thus

$$F = -\frac{\partial \alpha}{\partial n}\left[\cos \psi \left(\frac{\partial u}{\partial x} \cos \psi + \frac{\partial v}{\partial x} \sin \psi\right) + \sin \psi \left(\frac{\partial u}{\partial y} \cos \psi + \frac{\partial v}{\partial y} \sin \psi\right)\right]$$

or upon substituting from (108.4) and simplifying,

$$F = -\frac{\partial \alpha}{\partial n}(a \cos 2\psi + b) \qquad (108.5)$$

If the angle χ between the direction of α and the axis of dilatation is introduced,

$$\chi = 90° + \psi$$

$$F = \frac{\partial \alpha}{\partial n}(a \cos 2\chi - b) \qquad (108.51)$$

It will be noted that the frontogenetical effect depends only on the divergence b and on the deformation a, but not on the translatory motion and on the vorticity.

The sign of F is determined by the angle χ. F vanishes along a line given by the equation

$$\cos 2\chi' = \frac{b}{a} \tag{108.6}$$

$F > 0$ if at a given point $\chi' > \chi > -\chi'$, and $F < 0$ if

$$\pi - \chi' > \chi > \chi'.$$

In the first case, $\partial\alpha/\partial n$ increases, in the second case it decreases.

Thus, there are, in general, two symmetrical sectors in which $F > 0$ and two in which $F < 0$. These sectors are shown in

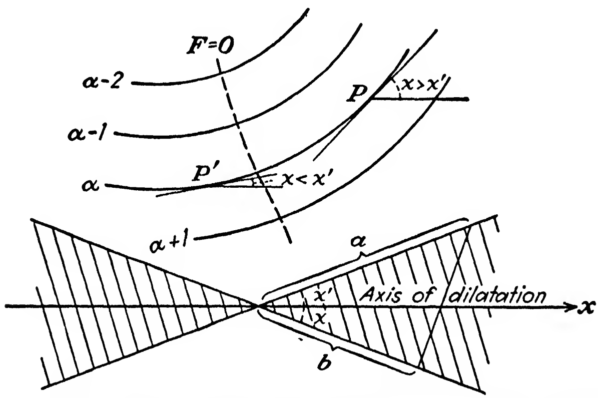

FIG. 74.—Frontogenetical and frontolytical sectors. (*After Petterssen.*)

Fig. 74. The two sectors whose boundary lines make the angle χ' with the positive and negative axes of dilatation are called the "frontogenetical sectors" (shaded in Fig. 74); the other two are called the "frontolytical sectors" by Petterssen. Consider the point P on the isopleth α. The angle χ between the tangent on α in P and the axis of dilatation is larger than χ'. The field is therefore frontolytical at P. At P', on the other hand, it is frontogenetical, for $\chi < \chi'$. Along the line $F = 0$, the angle between the tangents on the α-lines and the axis of dilatation is χ'.

When there is no divergence, $b = 0$, and $\chi' = 45°$. When the field of motion is convergent, $b < 0$, and $\chi' > 45°$; when it is

divergent, $b > 0$, and $\chi' < 45°$. When $b = a$, $\partial v/\partial y = 0$, $\chi' = 0$ and frontogenesis does not occur. Similarly, in the case $b > a$, which is theoretically possible, (108.6) cannot be satisfied; but it is seen from (108.51) that $F < 0$ everywhere.

For a practical application of these deductions, it is necessary to determine the divergence b and the deformation a of the field. In order to determine the line of frontogenesis in a given case, one has to determine along which line F has a maximum. For details and examples the reader is referred to Petterssen's publications cited previously.

109. The Wave Theory of Cyclones. When a frontal surface has developed in the atmosphere, wave motion may start on such a surface spontaneously, just as wave motions originate at the boundary between water and air.

It had already been suggested by Helmholtz,[1] although not in a very definite form, that cyclones, at least cyclones of extratropical latitudes, originate as small wave perturbations at surfaces of discontinuity. But the wave theory of the origin of cyclones has primarily been developed by V. Bjerknes and his collaborators from both the theoretical and the observational side.[2]

In order to show that the wave theory can explain the formation of cyclones, it must first be investigated whether waves of the right length and velocity originate in the atmosphere and whether such waves are unstable. In the case of unstable waves the amplitudes which are very small in the original wave perturbation increase with time until the cyclone loses its wave character and becomes a vortex like the fully developed cyclones found on the weather maps. The mathematical problems arising out of a study of these instability conditions are very difficult, and it cannot be claimed that a complete solution has been reached. But important results have been obtained which make it certain that unstable waves of the right length and velocity and with the correct fields of motion can originate at the atmospheric surfaces of discontinuity. A complete presentation of these investigations is beyond the scope of this book. But it is possible to show with the aid of the results of the preceding

[1] HELMHOLTZ, H., *Sitz.-Ber. Akad. Wiss. Berlin*, 647, 1888.
[2] BJERKNES, V., and collaborators, *op. cit.*

chapter that the existence of the postulated waves is highly plausible.

According to Sec. 102, waves of very short length will be unstable when the surface of separation between the two fluids represents also a wind discontinuity. Such a wind shear is, as a rule, found at frontal surfaces so that sufficiently small frontal waves will be unstable. With increasing wave length, however, the effect of gravitational stability becomes stronger and overcompensates the effect of shearing instability. It depends on the magnitude of the wind and temperature discontinuity at which wave length the transition from the unstable to the stable type occurs. The table on page 287 shows that the wave length of the longest unstable waves of the mixed gravitational and shearing type is less than 10 km at the temperature and wind discontinuities occurring in the atmosphere, so that the effects of shearing and gravitation alone cannot account for the formation of cyclone waves.

Actually, the limiting value of the wave length of the unstable waves is still smaller than would appear from the table on page 287, where the gravitational stability is due only to the heavier fluid being underneath the lighter one. In the atmosphere, however, each layer for itself is, at least as a rule, in stable equilibrium; for the actual lapse rate of temperature is smaller than the adiabatic. Thus, the gravitational stability of the atmosphere is really larger than the stability represented by the superposition of two incompressible fluid layers of different densities.

As longer waves are considered, the effect of the deflecting force of the earth's rotation becomes more important and the waves are now of a mixed shearing, gravitational, and inertia type. Of course, the inertia effect acts on short waves also; but its influence becomes noticeable only for longer waves. It was shown in Sec. 105 that the inertia effect in the earth's atmosphere acts as a stabilizer on the wave motion, for the angular momentum shows a stable distribution.

The deflecting force of the earth's rotation has, however, another important, more indirect effect on the stability of the waves. Because it acts perpendicular to the earth's axis it has a horizontal component everywhere except at the equator. This component causes an inclination of the perturbation motion to

the vertical as shown by (104.33) which represents the y-component of the perturbation motion. The inclination increases with the scale of the motion, and in long cyclonic waves the motion is predominantly horizontal. This tilting of the plane of the perturbation motion leads to the formation of unstable waves, in the following manner: The stable character of the longer waves of a mixed gravitational and shearing character is due to the difference in weight of the oscillating particle and its environment; thus, it depends on the amplitude of the vertical oscillation. As the wave motion becomes more horizontal with

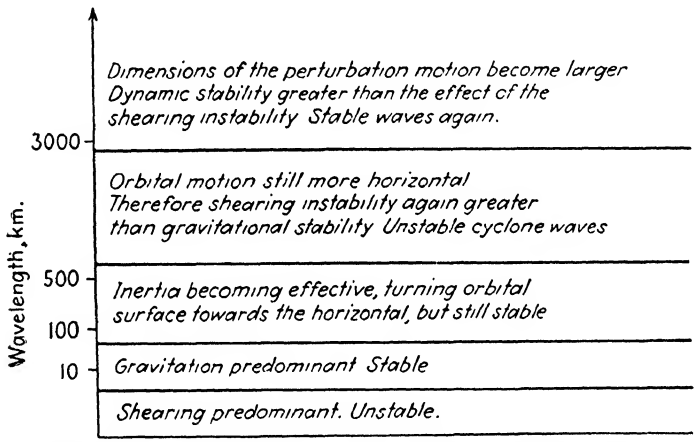

FIG. 75.—Schematic representation of the stability conditions of atmospheric waves. (The indicated wave lengths are very rough approximations and are not drawn to scale.)

increasing wave length, the stabilizing effect of gravitation decreases. Mathematical analysis shows that waves whose length is of the order of 500 km are unstable. The shearing instability for waves of these dimensions is greater than the gravitational and dynamic stability combined. With still longer waves, of about 3000 km, the effect of the stable distribution of angular momentum (dynamic stability, page 294) increases further, and thus still longer waves are stable. These considerations concerning the stability and instability of atmospheric waves are summarized in the preceding schematic table (Fig. 75). The cyclone waves must obviously be those types in which the shearing instability is larger than the gravitational stability owing to the tilting of the plane of the perturbation

motion by the Coriolis force. Thus, unstable waves of the right length to explain the formation of wave cyclones may form in the atmosphere. Because these waves are unstable, their amplitudes must increase until a fully developed cyclone is formed. The theory shows also that these waves have velocities of the same order of magnitude as the nascent cyclones, that the velocities are directed eastward, and that the motion of the air is of the type observed in nascent cyclones.

From the wave character of the cyclones, it follows that the velocity of a cyclone is not simply the mean of the velocities in the cold and in the warm current but that it depends also on a dynamic term (page 281) which represents the effects of gravitational stability, dynamic stability, and shearing instability. The wave theory of cyclones is sufficiently far advanced to enable us to assert that it has been proved that the formation of cyclones as waves at a frontal surface is possible. Nevertheless, a great number of difficult problems remain to be solved.

It was pointed out before that our qualitative considerations refer to incompressible fluids only. The gravitational stability of the atmosphere is greater than that of a fluid system consisting of two homogeneous layers of different densities. Solberg[1] has taken the compressibility into account by considering isothermal layers of air which follow the adiabatic law of compression and expansion. The assumption of an isothermal atmosphere makes mathematical analysis easier but implies that the gravitational stability is greater than in the actual atmosphere with its linear lapse rate of temperature. In consequence of this assumed greater gravitational stability the computed limit of instability lies at somewhat longer waves than in the actual atmosphere. But the assumption of isothermal layers does not change the main result, that unstable waves of the type of the observed cyclone waves can form in the atmosphere.

The frontal surfaces occurring in nature are, of course, not sharp mathematical discontinuities as assumed in the theoretical investigations but narrow zones of transition in which the properties of the air change rapidly but continuously. However, it has been shown that, in general, wave motions are practically the same whether there is a sharp discontinuity or a transitional zone, provided that the thickness of the transitional zone is small

[1] Bjerknes V., and collaborators, *op. cit.*, Chap. XIV.

compared with the wave length.[1] For cyclone waves, this condition is always fulfilled.

A somewhat more serious deficiency of the wave theory of cyclones which has not been mentioned so far is the assumption that the lower cold and the upper warm layer have boundaries parallel to the frontal surface (Fig. 76*a*). This assumption eliminates the mathematical difficulties that arise from the frontal surface intersecting the ground, and the cold air forming a wedge under the warmer air (Fig. 76*b*).

Solberg[2] has given a solution for a fluid system of the form shown in Fig. 76*b*. However, a discussion of this solution has not been undertaken owing to the mathematical difficulties.

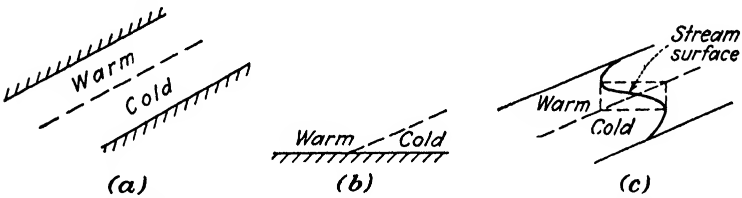

FIG. 76.—Solidification of stream surfaces. Position of the boundary: (*a*) in the theory, (*b*) in the atmosphere, (*c*) after solidification of the stream surface.

Another investigation of this problem was published by Kotschin,[3] who neglected the vertical accelerations and assumed that both the cold and the warm mass are bounded by a rigid upper surface. Kotschin was able to show that waves of the dimensions of the cyclones can be unstable under atmospheric conditions when the cold air lies in the form of a wedge under the warm air. However, Solberg has raised certain objections against the validity of Kotschin's work; for the vertical acceleration terms are omitted, and thus the problem cannot yet be considered as solved completely.

Another, somewhat circuitous, method of approach has been chosen by V. Bjerknes and Solberg.[4] They found during their investigations that some of the stream surfaces along which the motion of the air particles takes place are practically horizontal

[1] HAURWITZ, B., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **5**, 52–53, 73–74, 1932.

[2] SOLBERG, H., *Geofys. Pub.*, **5**, No. 9, 1928.

[3] KOTSCHIN, N., *Beitr. Phys. Atm.*, **18**, 129, 1928.

[4] BJERKNES, V., and collaborators, *op. cit.*, Chap. XIV.

for an extent of the dimensions of a cyclone, about 2000 km. Now, a stream surface can be assumed as rigid, for the fluid motion is parallel to it, by definition. Such a stream surface is shown in Fig. 76*c*. If this stream surface is solidified and regarded as the surface of the earth, a very close similarity is obtained between the conditions in the fluid system in the rectangle (Fig. 76*c*) above the solidified stream surface and the actual conditions in the atmosphere at a frontal surface. The cold air lies in the form of a wedge under the warm air, and the wave motion near the surface is almost parallel to the horizontal stream surface which may be identified with the earth's surface. Farther away from the surface of discontinuity the agreement with reality is, of course, less satisfactory, for the curvature of the solidified stream surface will be stronger. But mathematical analysis has shown that the intensity of the wave motion decreases exponentially with the distance from the surface of discontinuity, at least in the case of a compressible fluid of stable stratification such as the atmosphere. The field of motion is, therefore, dynamically most important at the frontal surface and loses its significance comparatively rapidly in lateral and vertical direction. Thus, the method of solidifying stream surfaces is more satisfactory than it might appear at first. This line of attack, however, can be regarded only as preliminary. It will be necessary, also, to study directly the problem of a wave motion at a frontal surface inclined to the ground, in order to obtain a final solution and to find reliable criteria for the stability or instability of observed waves.

The investigations thus far have dealt only with waves on a rotating plane. In view of the large dimensions of cyclones the spherical shape of the earth should be taken into account, too. An extension of the wave theory of cyclones appears necessary in this respect, although it is hardly to be expected that any fundamentally new results will be obtained when the earth is treated as a sphere.

110. Further Development of the Extratropical Cyclones. The Occlusion Process. When an unstable wave has developed at a frontal surface, the life history of the cyclone cannot be studied much longer by means of the perturbation method. The perturbation method assumes that the perturbations are small compared with the undisturbed motion. This assumption is no

longer justified when the unstable wave has attained the dimensions of a fully grown cyclone. The further cyclonic development is, of course, well-known from empirical studies, and a brief description will be given here. For more information the reader is referred to the textbooks on synoptic meteorology.

The front on which the cyclone wave develops runs as a rule in a west-easterly direction parallel to the isobars of the undisturbed current, but of course other directions occur, also. The west-east direction is preferred; for the horizontal pressure gradient is mainly in the meridional direction, and therefore the undisturbed geostrophic motion is from the west. The front separates a warmer southerly air mass with an eastward velocity from a colder northerly mass which moves either westward or

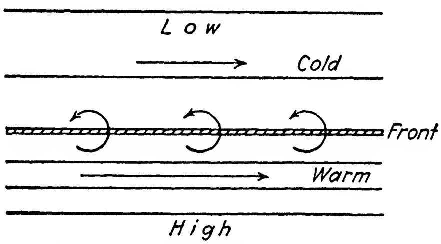

FIG. 77.—Initial, undisturbed state of cyclone development.

also eastward but with a smaller velocity than the warm mass (Fig. 77). The latter alternative occurs much more often, but in either case the vorticity at the frontal surface is cyclonic. When a wave perturbation occurs at the frontal surface, the shape of the front and of the isobars will become similar to that shown in Fig. 78*a*. The front is distorted into a wave and the isobars, or at least some of them, show a bulge toward the south, forming a region of somewhat lower pressure where the warm air has advanced northward. When this wave is unstable, the amplitude of the frontal distortion increases and the region of low pressure deepens (Fig. 78*b*). The front now forms a sector of warm air which is, in general, in the southern part of the cyclone. The stages represented by Fig. 78*a* and *b* are those to which the theoretical considerations of the preceding section apply. The cold air ahead of the warm sector is separated from the warm air by the warm front, and the cold air behind the warm sector by the cold front. The vertical cross section through the cyclone

south of the center shows the two wedges of cold air between which the warm air is situated.

The warm air ascends over the cold air along the warm-front surface, and thus cloud and rain are formed ahead of the warm front. The wedge of cold air behind the cold front lifts the warmer air, and rain occurs behind the cold front, also. The

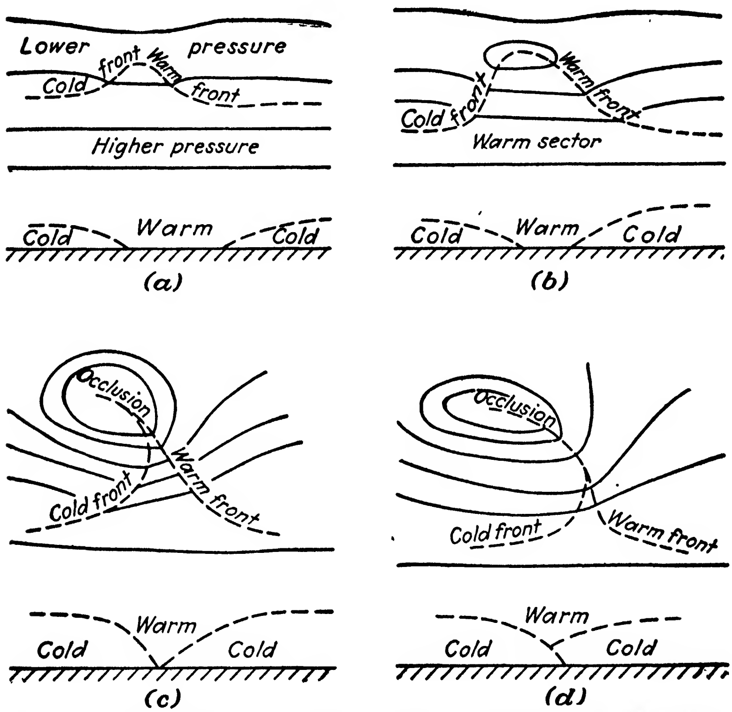

FIG. 78.—Development of a wave cyclone. Upper figures: isobars (full curves) and fronts (dotted) at the surface. Lower figures: vertical cross sections south of the cyclone center.

resulting distribution of the rain areas are shown schematically in Fig. 79. Experience has shown that, in the wedges of cold air, subsidence and a simultaneous spreading of the air take place so that the areas covered by cold air increase while the warm sector becomes smaller and smaller. Finally, the cold front catches up with the warm front, at first near the center of the cyclone, then farther away from it. This process is called the *occlusion process* of the cyclone, and the part of the front where

the cold front has caught up is called the *occlusion*. The beginning of the occlusion process is shown in Fig. 78*c*, and a more advanced stage in Fig. 78*d*. During the occlusion process the warm air is gradually lifted over the cold wedges as shown in the lower part of Fig. 78*c* and *d*. The juxtaposition of cold and warm air represents a certain amount of energy, as explained in Chap. XII. As the warm mass is lifted, the amount of internal and potential energy is dimin shed and transformed into other forms of energy, mainly kinetic, to maintain the circulation of the cyclone. When the occlusion process is completed, the cyclone represents a vortex with symmetrical temperature distribution which finally disappears owing to the effects of eddy and surface friction.

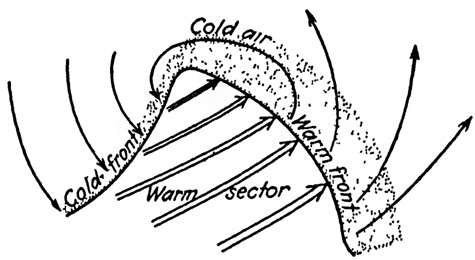

FIG. 79.—Rain areas in a fully developed cyclone. (*After J. Bjerknes and Solberg.*)

Occasionally, very cold air is introduced into the occluded cyclone. Then the old cold air, which has been in the cyclonic circulation for a long time and which has become relatively warm, acts as a new warm sector. In this situation the occluded cyclone can increase its kinetic energy again. This process is referred to as "regeneration" of the cyclone. Such a regenerated cyclone has been investigated by Schröder[1] with the aid of Margules's formulas for atmospheric energy transformations (Chap. XII).

As a rule, cyclones appear not singly but in groups of about three to six (Fig. 80), as shown by Bjerknes and Solberg.[2] From the wave nature of the nascent cyclone, it is easily understandable that not only one but a number of waves develop at a frontal surface. The first of these cyclones travels farthest to the north,

[1] SCHRÖDER, R., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **4**, 49, 1929.
[2] BJERKNES, J., and SOLBERG, H., *Geofys. Pub.*, **3**, No. 1, 1922.

and each successive cyclone is younger and appears farther to the south. The front is thus displaced farther south. Such a cyclone series is called a cyclone "family." It appears most regularly over northern oceans in winter. Finally, the individual cyclone family is terminated by an outbreak of the polar air extending far to the south.

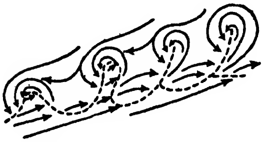

FIG. 80.—Cyclone family. (*After J. Bjerknes and Solberg.*)

Some of the ideas suggested by the wave theory of cyclones had already been proposed in similar form in earlier papers, but the Norwegian school arrived at their theory mainly independently and were the first to formulate a well-developed theory based on observations and on the principles of mathematical physics.

About the middle of the last century, Dove[1] found that the weather in the temperate latitudes is dominated by the motions

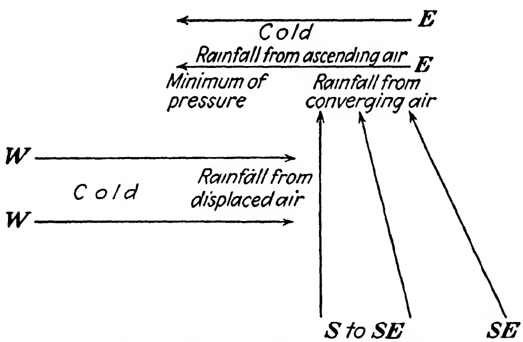

FIG. 81.—Cyclone model. (*After Shaw.*)

of polar and equatorial currents whose essential difference is their temperature. A still greater similarity to the principles of the polar-front theory is shown in the publications of Blasius,[2] who explains the precipitation on the basis of the surfaces of discontinuity—as they would be called today—between the polar

[1] DOVE, W. H., "Meteorologische Untersuchungen," Sander'sche Buchhandlung, Berlin, 1837.

[2] BLASIUS, W., "Storms, Their Nature, Classifications and Laws," Porter and Coates, Philadelphia, 1875. See also H. VON Ficker, *Sitz.-Ber. preuss. Akad. Wiss., Phys.-Math. Kl.*, p. 248, 1927.

and equatorial air masses. Bigelow[1] ventured the opinion that cyclones may be produced as vortices when a cold northerly and a warm southerly current meet. Finally, the reader is referred to Fig. 81 which represents a cyclone model published by Shaw[2] in 1913. In this schematic drawing many of the essential features of the later cyclone model of Bjerknes are found.

111. The Barrier Theory. Besides the wave theory of cyclogenesis a number of other theories have been advanced to explain the origin of cyclones.

According to Exner's hypothesis,[3] the existence of a temperature discontinuity in the air is regarded as a condition necessary for the formation of a cyclone, as in the wave theory. He considers an outbreak of polar air where a cold mass from the polar region intrudes in the region of westerly winds in the temperate latitudes, as shown in Fig. 82*a*. The cold air moves either westward or much more slowly eastward than the warmer air of the temperate latitudes. It acts as a barrier to the motion of the warm air, and therefore Exner's theory is referred to as the *barrier theory*. Because the faster moving warm air to the east of the cold tongue retains its velocity owing to inertia, the pressure is here reduced considerably. The formation of the region of low pressure changes the direction of the air motion at A, B, and C, as indicated in Fig. 82*b*, so that a cyclonic vortex develops. Because, the cold tongue C forms part of this cyclonic vortex, its direction of motion becomes also mainly eastward.

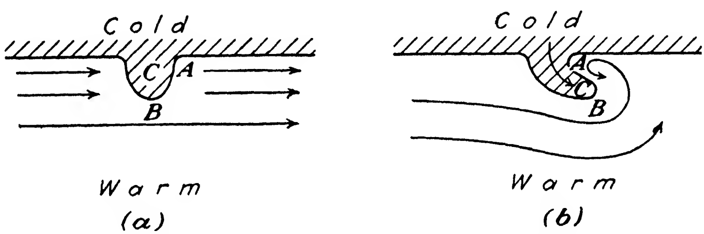

FIG. 82.—Exner's barrier theory of cyclogenesis.

[1] BIGELOW, H., *Monthly Weather Rev.*, **30**, 251, 1902.

[2] SHAW, N., "Forecasting Weather," Constable & Company, Ltd., London, 1913.

[3] EXNER, F. M., "Dynamische Meteorologie," 2d ed., p. 339, Verlag Julius Springer, Vienna, 1925.

In a similar manner a continent, especially a mountainous continent, may act as a barrier. Exner refers in particular to Greenland and suggests that the Icelandic minimum to the east of the southern tip of Greenland may originate as a barrier effect.

However, it appears doubtful that the pressure deficit in the lee of the barrier is strong enough to lead to cyclogenesis except in very special cases. A mathematical analysis of Exner's theory that would give a reliable estimate of the pressure deficit has not yet been undertaken. The calculations on page 240 take into account only the static pressure effect, not the pressure decrease due to divergence behind the barrier.

112. The Convection Theory. The simultaneous existence of a cold and a warm current of air is regarded as necessary for cyclogenesis according both to the wave theory and to the barrier theory. The *convection theory* suggests an entirely different mechanism. When the air over a limited portion of the earth is heated until its temperature is higher than that of the surrounding air, it will begin to rise. As long as it is unsaturated, it cools at the dry-adiabatic rate of 1°C/100 m. The lapse rate of the surrounding air is, in general, considerably smaller than the dry-adiabatic. The air, therefore, cannot rise to very high levels, for it will soon reach equilibrium with its surroundings. For instance, if its temperature is originally 12°C higher than the temperature of the environment and if the lapse rate in the surrounding air is 6°C/km, it can reach an altitude of only 3 km. There the temperature of the ascending and the surrounding air become equal. If saturation is reached during the ascent, however, the rate of cooling is diminished considerably, and the air may rise to much greater heights. The importance of the moisture content of the air for the formation and maintenance of cyclones was emphasized especially by Refsdal,[1] although from a different point of view.

The mass deficit created by the ascent of air must be compensated by the inflow of air toward the regions over which the ascent takes place. When such convergence toward a center takes place on the rotating earth, the air acquires a cyclonic vorticity, as follows directly from (85.2) (see also page 293).

[1] Refsdal, A., *Geofys. Pub.*, 5, No. 12, 1930.

The same result can also be derived in more elementary fashion from the principle of the conservation of angular momentum[1] (see Sec. 44). If a particle of air is at first at rest with respect to the earth at a distance r' from the center of convergence, its momentum is $r'^2\omega \sin \varphi$ where ω is the angular velocity of the earth's rotation and φ the geographic latitude. When the particle arrives at r, it has acquired a tangential velocity component v relative to the earth, and its angular momentum is now

$$vr + r^2\omega \sin \varphi$$

Upon equating both expressions for the angular momentum, it follows that

$$v = \left(\frac{r'^2}{r} - r\right) \omega \sin \varphi \tag{112.1}$$

Convergence, $r' > r$, therefore produces cyclonic rotation, and divergence anticyclonic rotation. This is also true for the Southern Hemisphere, for there the direction of cyclonic and anticyclonic rotation is opposite to the direction in the Northern Hemisphere.

We shall assume that the ascent due to convection takes place only in a central core while the outer part of the air column shrinks, so that its outer radius decreases from R_0 to R (Fig. 83). Then an inner circle with radius r' outside the core consisting of fluid particles will shrink to r. Because the mass in the annular region between R_0 and r' must remain constant,

$$\pi(R_0^2 - r'^2) = \pi(R^2 - r^2)$$

or

$$r'^2 - r^2 = R_0^2 - R^2$$

Upon substituting in (112.1), it follows that

$$v = \frac{R_0^2 - R^2}{r} \omega \sin \varphi \tag{112.2}$$

The resulting velocity distribution in the outer part is inversely proportional to the distance from the center. But in the central core a different law holds for the velocity distribution, and the velocity does not become infinite at the center.

[1] See, for instance, D. Brunt, "Physical and Dynamical Meteorology," 2d ed., p. 300, Cambridge University Press, London, 1939.

The cyclone is a region in which the pressure is lower than in the surrounding air. Therefore, the air that converges toward the center and ascends must be carried away at greater altitudes. The air may be removed by an upper current with a velocity and direction of motion different from that of the air flow in the lower atmosphere.

However, it seems unlikely that an appreciable number of extratropical cyclones are caused by convection and convergence. It is difficult to visualize a sufficiently strong heating in temperate latitudes, especially during the winter when the cyclones are most frequent. Furthermore, most extratropical cyclones are associated with frontal systems even in their earliest stages. The convection theory seems more promising as an explanation

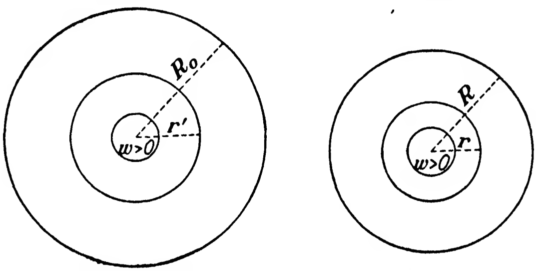

FIG. 83.—Horizontal shrinking of an air column and the generation of rotation.

of the origin of tropical cyclones, which will be discussed in Sec. 115.

On the other hand, it is not entirely impossible that different cyclones have different modes of origin; thus, some may actually represent homogeneous vortices at their beginning, not only after the occlusion process is completed. These cyclones would then belong to a different category from the wave cyclones. The only common characteristics would be the center of low pressure, and the cyclonic direction of motion.

113. The Conditions in the Upper Levels. The theories of cyclogenesis discussed in the preceding sections do not take into account the processes in the higher levels of the troposphere and in the stratosphere. It is *a priori* plausible that a close connection must exist between the variations in the upper and lower atmospheric levels over pressure systems. The pressure at any altitude represents with a great degree of accuracy the weight per unit area of the total air above the level. Consequently, the

pressure variations at any height represent the total effect of the mass variations in the higher layers. The theoretical foundations for the discussion of these variations are contained in Eq. (58.3). The last term of this equation shows how the lower layers can affect the pressure at a higher level when vertical motions occur. The relation between pressure and temperature changes has been discussed already in Secs. 11 and 12.

W. H. Dines[1] has expressed the connection between different variables by a number of correlation coefficients[2] which are partly given in the following table. The data are based on observations over Europe. The notation is as follows:

P_0 = pressure at mean sea level
P_9 = pressure at 9 km
T_0 = temperature at surface
T_{0-4} = mean temperature from surface to 4 km
T_m = mean temperature from 1 to 9 km
H_c = height of tropopause
T_c = temperature at tropopause

CORRELATION COEFFICIENTS
(After W. H. Dines)

	P_0	P_9	T_0	T_{0-4}	T_m	H_c	T_c
P_0	......	0.68	0.16	0 34	0.47	0.68	−0.52
P_9	0.68	......	0.28	0.28	0.95	0.84	−0.47
T_0	0.16	0.28	....	....	.....	0.30	
T_{0-4}	0.34	......	....	....	......	0.66	
T_m	0.47	0.95	....	....	......	0.79	−0.37
H_c	0.68	0.84	0.30	0.66	0.79	......	−0.68
T_c	−0.52	−0.47	....	....	−0.37	−0.68	

The surface pressure shows very little correlation with the surface temperature, for the upper layers contribute also to the surface-pressure variation. Even the correlation between the surface pressure and the mean temperature up to 9 km is not very strong, but it is greater than the correlation with the mean temperature of the layers up to 4 km. More clearly pronounced

[1] DINES, W. H., *Geophys. Mem.*, No. 13, p. 67, 1919.

[2] If x is the deviation of each member of one series of observations from the mean value and y the same for the other series, the correlation coefficient is $\Sigma(xy)/\sqrt{\Sigma x^2 \Sigma y^2}$. It lies between +1 and −1.

is the correlation between the surface pressure and the pressure at 9 km. Equally large is the correlation with the height of the tropopause. T_c denotes the temperature at the tropopause, that is at a variable height. In general, the temperature at the tropopause increases as the height of the tropopause decreases, and vice versa, as indicated by the negative correlation coefficient between H_c and T_c. Because the correlation between P_0 and H_c is positive, the correlation between P_0 and T_c is negative.

Of the other correlations the one between P_9 and T_m deserves special attention for its magnitude. When the pressure at 9 km is high, the temperature in the layer between the earth's surface and 9 km elevation is also high. When the pressure at this altitude is low, the mean temperature in the layer up to 9 km is also below normal. A close connection exists, also, between P_9 and H_c and between H_c and T_m. All these correlations are positive, and the deviations from the mean are thus all in the same direction.

Similar, although somewhat smaller, correlation coefficients have been computed by Schedler,[1] also from the European observations. The correlations between the meteorological variables over North America are of the same magnitude.[2] In order to obtain a more detailed picture of the simultaneous and subsequent variations of pressure and temperature at different levels during the passage of centers of high and low pressure at the ground, Penner[3] divided each pressure center into three regions, the east side, the center, and the west side, following a method first used by Schedler[4] and later by Haurwitz.[1] Because the centers move as a rule from west to east, these regions may be called the "front," the "center," and the "rear" of the low and of the high. This notation indicates the order in which the regions pass over the point of observation. The available aerological observations for the investigated station are then arranged in six groups according to the position of the observing station. For each group the mean value of the pressure and temperature at various levels is computed. The curves obtained

[1] Schedler, A., *Beitr. Phys. Atm.*, **7**, 88, 1917.

[2] Haurwitz, B., and Turnbull, W. E., *Can. Met. Mem.*, **1**, 3, 1938. Haurwitz, B., and Haurwitz E., *Harvard Met. Studies*, No. 3, 1939.

[3] Penner, C. M., *Can. J. Res.*, **A19**, 1, 1941.

[4] Schedler, A., *Beitr. Phys. Atm.*, **9**, 181, 1921.

by plotting these data in the order in which the regions pass over the point of observation may be considered as mean barograms and thermograms at different levels. Penner has used radio soundings made at Sault Sainte Marie, Mich., for the construction of these curves. The conditions over Europe and over North America are quite similar as can be seen from a comparison of the results of the investigations of Schedler and

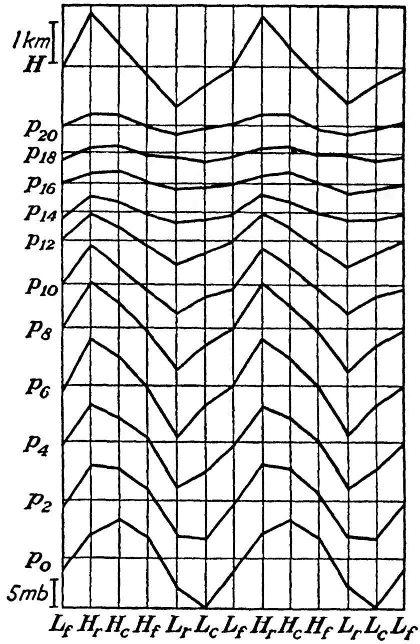

FIG. 84.—Pressure variation at different altitudes over Sault Sainte Marie during the passage of highs and lows. L_f, L_c, L_r = front, center, and rear of a low; H_f, H_c, H_r of a high. H = height of the tropopause. (*After Penner.*)

Haurwitz, respectively. Penner's curves, which are reproduced in Figs. 84 and 85, may therefore be regarded as representative for north-temperate latitudes in general.

The curves of Fig. 84 show clearly that the pressure extremes are progressively retarded with increasing altitude in the troposphere. Above 12 km, however, in the stratosphere, the retardation decreases again so that the position of the maxima and minima becomes more nearly equal to that of the surface pressure. The temperature extremes (Fig. 85), on the other hand, occur

earlier at higher levels throughout the troposphere. At the 10-km level the temperature curve is very irregular. Above this level the phase of the temperature curve is opposite to the phase in the troposphere. The minima of the temperature in the stratosphere take place approximately over the region of maximum temperature in the lower layers, and vice versa. It appears, thus, that the temperature variations at 10 km some-

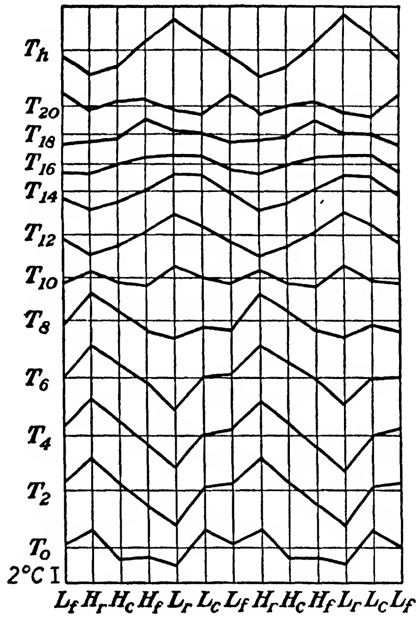

FIG. 85.—Temperature variations at different altitudes over Sault Sainte Marie during the passage of highs and lows. L_f, L_c, L_r = front, center, and rear of a low; H_f, H_c, H_r of a high. T_h = temperature at the tropopause. (*After Penner.*)

times follow the pattern of the tropospheric temperature variations and at other times the pattern of the stratospheric temperature variations. The variation of the height of the tropopause (Fig. 84) follows very closely the variation of the pressure at the levels near the tropopause, as is indicated by the correlation coefficient 0.84 between P_9 and H_c, according to Dines. The temperature at the height of the tropopause, *i.e.*, at a variable height, runs parallel to the temperature at 12 km and opposite to the height of the tropopause, as might be expected.

The results that are expressed in Dines's correlation coefficients and in Penner's curves are based on statistical investigations of a great number of observations. Such a statistical analysis has the advantage that it shows the average course of events in the atmosphere, whereas the study of a particular weather situation cannot reveal whether this situation is typical or not. The study of a great number of individual cases is necessary before one can arrive at a conclusion about the relative frequency of different cases. On the other hand, a statistical investigation must necessarily smooth out many of the differences between the individual cases. This may lead to an oversimplification of the picture. A corroboration and a refinement of the statistical results by the investigation of individual cases are therefore obviously necessary.

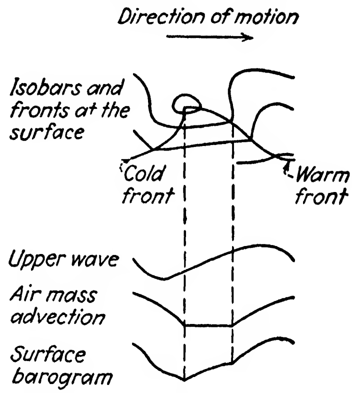

FIG. 86.—The surface-pressure variation during the passage of a young cyclone. (*After J. Bjerknes.*)

The number of such investigations is not yet so great as would appear desirable, owing to the difficulties in obtaining sufficient data from the upper air. Nevertheless, these studies, and especially the ones by J. Bjerknes and Palmén,[1] have shown that the statistical picture of the relations between the pressure and temperature variations in the upper and the lower atmosphere is largely correct.

We shall briefly discuss in which manner the statistical results fit into the picture of a wave cyclone. Figure 86 shows, after Bjerknes, how the variation of the surface pressure during the passage of a young cyclone is brought about by the changes due to the passage of cold and warm masses and by the upper pressure wave. The upper part of the figure gives the distribution of the isobars and fronts at the surface. The broken straight line indicates the successive positions of the station for which the

[1] See, for instance, J. Bjerknes, and E. Palmén, *Geofys. Pub.*, **12**, No. 2, 1937; E. PALMÉN, *Soc. Sci. Fenn. Comm. Phys.-Math.*, **7**, 6, 1933. Some of the results mentioned below have been presented in a course of lectures by Dr. J. Bjerknes at the Meteorological Office, Toronto, in August, 1939.

barogram is derived. The effect of the warm and cold front on the surface pressure is shown by the curve called "air-mass advection," which can be computed according to Sec. 65. The pressure falls under the warm-front surface until the frontal passage occurs at the ground. While the station is in the warm sector, the pressure would remain constant. With the arrival of the cold front the pressure starts to rise again. Upon the effect of the air-mass advection the upper pressure wave must be superimposed, which represents the influence of the mass variations of the upper layers on the surface-pressure variation. This is obtained by deducting the pressure effect of the polar-front advection from the surface barogram. The maximum of the upper pressure wave occurs over the warm-front surface. This agrees with the statistical results shown in Figs. 84 and 85 according to which the temperature in the upper troposphere is high in the rear of an anticyclone, the same area over which the upper pressure reaches a maximum. The minimum of the upper pressure wave is placed over the cold air behind the cold front, which agrees also with the statistical results, since Figs. 84 and 85 show that the lowest pressure in the upper troposphere is situated over the rear of a low where the temperature of the upper troposphere is also lower. The result of the superposition of the upper wave and the pressure effect of the polar-front advection is shown in the lowest curve of Fig. 86, marked "surface barogram." The surface pressure is falling throughout the warm sector until the arrival of the cold front, when it attains its minimum. At higher levels the pressure continues to fall so that the pressure minimum occurs here later.

In occluded cyclones the effect of the air-mass advection is not so strong. The upper wave, on the other hand, is more strongly developed over old occluded cyclones, and the retardation of the minimum pressure with respect to the surface is therefore less. In the statistical data of Figs. 84 and 85 the conditions in young and old cyclones are, of course, not separated.

In young cyclones the isobars in the upper troposphere run mainly from west to east and the disturbance that appears as a frontal cyclone at the ground is marked only by a wavelike elongation of the isobars in northerly and southerly direction. The tropopause height shows only a slight variation. As the cyclone grows older and becomes occluded, the upper pressure

field deepens and some of the isobars may even become closed. Simultaneously the height variation of the tropopause increases considerably.

Palmén[1] has suggested that the large height variation of the tropopause may not always be due to an actual motion of this surface. He assumes, instead, that during the vertical motion the tropopause is dissolved in certain regions while it is formed in others. It follows from Eq. (10.21) that the lapse rate decreases when a layer of air is lowered, whereas it increases if the layer is lifted adiabatically. If the tropopause is defined as the surface where the vertical lapse rate of temperature first decreases below a certain value, it would then be necessary to regard the tropopause as being at a different level. This implies that over regions with strong vertical motions the tropopause will not appear as a continuous surface, but rather in a discontinuous form as shown in Fig. 87. The full thick curves represent the tropopause; the broken thick curves indicate heights at which the temperature-height curve of an aerological ascent would show a marked change of the vertical lapse rate of temperature which, however, would not be regarded as the tropopause according to the customary definition. The thin broken curve gives the position of the tropopause if the possibility of its dissolution and re-formation at another level is overlooked. Such cases of a multiple tropopause have been reported by J. Bjerknes,[2] Palmén,[2] Mieghem,[3] and others.

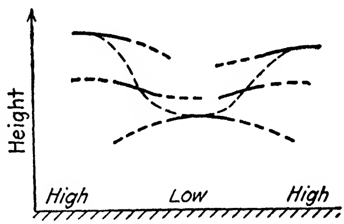

FIG. 87.—Multiple tropopause. (*After Palmén.*)

114. The Theories of the Coupling between the Variations in the Higher and the Lower Atmosphere. The various hypotheses concerning the coupling between the variations in the upper and the lower atmosphere may be divided into two groups. In one the variations in the upper atmosphere, at the tropopause, and in the stratosphere are regarded as the cause of the variations below; in the other, the atmospheric processes in the lower

[1] PALMÉN, *loc. cit.*

[2] BJERKNES, J., and PALMÉN, *loc. cit.*

[3] VAN MIEGHEM, J., *Mém. inst. roy. Belg.*, Vol. 10, 1939.

troposphere are considered as the cause of the variations above.

Stüve's "thermocyclogenesis"[1] may be taken as typical for the first group. Stüve starts out from the fact that the pressure in the stratosphere decreases toward the pole (see Fig. 67). If the tropospheric mean temperature T_{tr} were constant in the meridional direction, the horizontal pressure gradient in the stratosphere would give rise to a very large pressure gradient at the surface as follows immediately from the barometric equation (6.21) by differentiation with respect to the latitude,

$$\frac{1}{p_0}\frac{\partial p_0}{\partial \varphi} = \frac{1}{p}\frac{\partial p}{\partial \varphi} - \frac{gz}{RT_{tr}^2}\frac{\partial T_{tr}}{\partial \varphi}$$

As long as T_{tr} is constant, the ratio of the pressure gradients at the surface and in the stratosphere is equal to the ratio of the

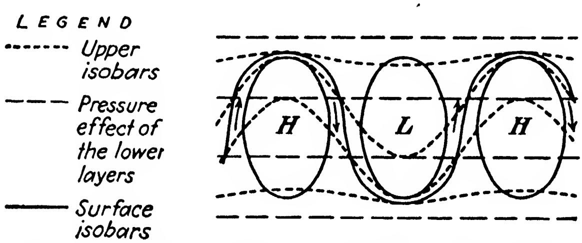

FIG. 88.—First stage of thermocyclogenesis. (*After Stüve.*)

pressures at these levels. The decrease of the tropospheric mean temperature northward, however, partly compensates the stratospheric pressure effect (note that $\partial p_0/\partial \varphi$ and $\partial p/\partial \varphi$ are negative), so that the surface-pressure gradient in the meridional direction is much less strong than in an atmosphere without meridional temperature gradient. Stüve considers first the effect of meridional oscillations of the air in the stratosphere, alternately northward and southward (Fig. 88, dotted lines). The field of pressure in the stratosphere has now a wedge of high pressure where the air has been displaced northward and a trough of low pressure where the air has been displaced southward.

The pressure effect of the lower tropospheric layers increases toward the north. Upon adding this to the stratospheric pressure distribution, centers of low pressure are formed at sea level over the areas where the stratospheric air moves south and

[1] STÜVE, G., *Beitr. Phys. Atm.*, **13**, 23, 1926.

centers of high pressure where the stratospheric air moves north (Fig. 88, full curves). As the surface-pressure centers are formed, the air in the troposphere is set in motion, too. According to the geostrophic wind relation, which is assumed to hold, the tropospheric air moves in a southerly direction on the west side of a low and on the east side of a high and in a northerly direction on the west side of a high and on the east side of a low. The southward motion of colder tropospheric air increases the tropospheric pressure effect on the west side of the low and on the east side of the high, and thus the center of the surface high moves toward the east. Similarly, the surface low is displaced forward with respect to the upper trough of low pressure by the current of warmer air on the east side of the low and on the west side of the high. If these pressure systems pass over a given station, the extreme values of the surface pressures will precede the extremes at greater heights, as is actually observed. The forward displacement of the pressure extremes at greater heights in the stratosphere found by Penner (Fig. 84) was not known when Stüve developed his hypothesis of thermo-cyclogenesis and is not taken into account although his hypothesis may conceivably be extended to cover this phenomenon.

The forward displacement of the temperature extremes in the higher troposphere is not so easily explained as the retardation of the pressure extremes. If the stratospheric troughs and wedges are the cause, the surface lows and highs the secondary effect, the stratospheric motions, and therefore the temperature changes, in the stratosphere will occur earlier than the temperature changes in the lower atmosphere. But the matter is complicated by the fact that the motion in the lower layers must begin simultaneously with the upper motion, not only after the latter have caused an appreciable deformation of the upper temperature and pressure fields.

Stüve arrived at his hypothesis by graphic addition of each of the successive stratospheric pressure fields to the corresponding tropospheric pressure effect. His conclusions are based on the barometric formula and on the geostrophic wind relation. If the latter assumption were to hold exactly, no local pressure variations could occur at the earth's surface (page 160); thus, Stüve's theory can hardly give a satisfactory interpretation of the relation between the upper and lower atmospheric layers.

The explanation of the pressure variations is a dynamic problem. It seems impossible to solve such a dynamic problem by the barometric equation, which is taken from atmospheric statics, and by the geostrophic relation which assumes a balance between the Coriolis and pressure-gradient force.

The concept according to which the atmospheric disturbances in the lower levels are caused by variations in the upper atmosphere is closely connected with the theory of "stratospheric steering," in which it is assumed that the motion of the surface

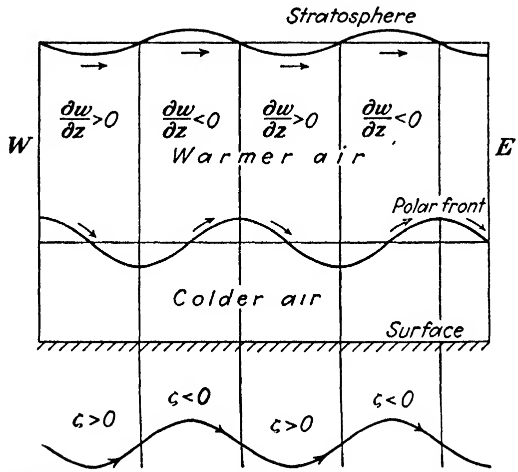

FIG. 89.—Coupling between polar front waves and tropopause waves in the initial stage. (*After J. Bjerknes.*)

pressure is steered by the pressure distribution at high levels, especially in the stratosphere.

A scheme of atmospheric development that starts out from a consideration of the variations in the lower atmosphere has been developed by J. Bjerknes.[1] In the absence of perturbations the polar front can be regarded as a plane ascending northward. A vertical cross section from west to east through the atmosphere north of the polar front at the surface shows the frontal surface as a horizontal straight line with the warmer air above and the colder air below. The tropopause whose height decreases northward appears likewise as a horizontal straight line. When wave motion begins at the polar front, it appears as a sinusoidal

[1] BJERKNES, J., and collaborators, *op. cit.*, pp. 741–755.

line in the cross section (Fig. 89). Because the warmer air aloft moves faster than the colder air below, the warm air ascends on the westerly side of the wave crests and descends on the easterly side. The amplitude of this forced vertical motion of the warmer air decreases with the altitude, as follows from its nature as a wave motion and as is borne out by the observations. The vertical motion decreases consequently with the altitude, $\partial w/\partial z < 0$, over the west side of the wave crests and increases with the altitude, $\partial w/\partial z > 0$, over the east side; for w is here negative, and its absolute value decreases with the height.

This gives rise to cyclonic and anticyclonic vorticity according to Eq. (85.2)

$$\frac{d}{dt}(\zeta + 2\omega \sin \varphi) = -(\zeta + 2\omega \sin \varphi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{85.2}$$

The Coriolis parameter may here be regarded as constant, and the fluid may be assumed as incompressible; therefore, according to the equation of continuity (47.3),

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Then

$$\frac{d\zeta}{dt} = (2\omega \sin \varphi + \zeta)\frac{\partial w}{\partial z} \tag{114.1}$$

At first, $\zeta = 0$. Because $\varphi > 0$ in the Northern Hemisphere, $d\zeta/dt > 0$, and cyclonic rotation develops according to (114.1) where $\partial w/\partial z > 0$; $d\zeta/dt < 0$ and anticyclonic rotation develops where $\partial w/\partial z < 0$. For the Southern Hemisphere the same result holds. The horizontal streamlines that result thus and their positions relative to the troughs and crests of the frontal surface are shown in the lowest part of Fig. 89. Consequently, meridional oscillations are superimposed on the west-easterly motion of the warm air. The meridional oscillations extend into the stratosphere. The height of the tropopause and the pressure in the stratosphere decrease northward. Therefore, the tropopause and the stratospheric pressure are lowest where the air has moved farthest south and highest where the air has moved farthest north. Under these circumstances, the wave crests and troughs of the tropopause do not coincide with the wave

crests and troughs of the polar front (Fig. 89). The latter may be identified with the high and low pressure at the earth's surface. Thus, the retardation of the height of the maxima and minima of the tropopause and of the pressure extremes in the lower stratosphere with respect to the surface pressure (Fig. 84) can be accounted for.

With the beginning of the wave motion in the warm air, the boundaries of the regions where $\partial w/\partial z > 0$ and $\partial w/\partial z < 0$ no longer coincide exactly with the crests and troughs of the frontal wave but are displaced westward. This westward displacement is compensated, however, by the higher eastward velocity of the warm air than of the cold air. The regions of cyclonic and anticyclonic vorticity move toward the east of the positions that they occupied in the initial stage. Therefore, the phase difference between the tropopause wave and the polar-front wave which would be 90° according to Fig. 89 becomes smaller and changes with time. The variation of the phase difference is important for the deepening of the cyclone. The closer together the upper and lower pressure minimum occur the deeper the low at the ground will become, as is easily seen from Fig. 86. Frequently, the upper wave persists for some time after the occlusion of the cyclone; thus, on the weather map, a cyclone without frontal structure is observed.

The displacement of the temperature extremes forward with altitude may be explained by the increasing wind velocity with the altitude. The completely opposite course of the temperature variations in the troposphere and the stratosphere would appear to be a consequence of the reversal of the meridional temperature gradient at the transition from the troposphere to the stratosphere. In the troposphere, advection from the north brings colder air, in the stratosphere, warmer air; advection from the south brings warmer air in the troposphere, colder air in the stratosphere.

Both theories, the theory of stratospheric steering and the polar-front theory, lead to about the same conclusions concerning the coupling between the lower and the upper layers. This is to be expected, for the variations at one level imply variations at other levels which must start practically simultaneously. Consequently, the atmosphere cannot be segregated into different parts. No particular layer can assume a predominant role.

Schmiedel[1] refers in this connection to the investigations of Weickmann and his collaborators who have shown that very frequently the pressure variations (and therefore the weather) are dominated by periodic variations. These variations must be interpreted as oscillations of the whole atmosphere, with simultaneous wave motions at the polar front and at the tropopause.

So far in the explanation of the interaction between the different atmospheric layers, only the effects of the meridional displacements have been considered. It is, however, highly probable that vertical motions contribute a considerable share to the upper pressure variations. Meridional displacements of air could produce only wedges of high and troughs of low but not closed pressure centers which are at least sometimes observed in the upper atmosphere. From Eq. (58.1), it follows that owing to vertical motion alone,

$$\left(\frac{\partial p}{\partial t}\right)_h = (g\rho w)_h$$

Ascending motion through the level h brings, therefore, an increase of the pressure at h, and descending motion a fall of the pressure. Such vertical motions have been suggested by Palmén (see page 327) as an explanation for the multiple tropopause. It is quite possible that the low altitude of the tropopause behind the surface center of low and its high altitude behind the surface center of high are in part due to descending motion in the rear of the cyclone and to ascending motion in the rear of the anticyclone. The effect of vertical motion on the pressure can be quite large. The density at 10 km is approximately 0.4×10^{-3} gm/cm^3. If the vertical velocity is 1 cm/sec, the pressure variation would be 4 mb/3 hr. It is therefore very likely that vertical motions contribute considerably to the observed pressure variations at the tropopause.

A completely satisfactory theory of the coupling between the oscillations of the polar front and of the tropopause would have to treat the atmosphere as consisting of three layers, the cold air lying in the form of a wedge on the surface, the warm air separated from the cold air by the polar-front surface, and the

[1] Schmiedel, K., *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **9**, 1, 1937.

stratosphere separated by the tropopause from the lower atmosphere. Such a model of the structure of the atmosphere though still rather schematic is more general than the two-layer system generally considered in the wave theory of cyclones (Sec. 109). But even this simpler two-layer problem has not yet been completely solved, owing to its mathematical difficulty. It is therefore hardly to be expected that a complete solution of the three-layer problem will be forthcoming soon, although some progress may be made in this direction.

115. Tropical Cyclones. In tropical cyclones the wind velocity is much higher than in the extratropical cyclones of temperate latitudes. The pressure at the center of the tropical cyclones reaches sometimes very low values, 920 mb and less; but similarly low values have also been observed in temperate latitudes, according to Hann and Süring.[1] The characteristic difference between tropical and extratropical cyclones is the pressure gradient which is much steeper in tropical cyclones, for their diameter is considerably smaller than the diameter of extratropical cyclones.

Tropical cyclones rarely form closer to the equator than at 5 to 6° latitude, which indicates that the horizontal component of the Coriolis force is an important factor in their development. It seems that the tropical cyclones develop over the sea only.

The tropical cyclones move generally in a westerly to northwesterly direction in the Northern Hemisphere. Many assume a north-easterly or northerly direction at latitude 20 to 25°, and thus their track resembles a parabola. When they reach higher latitudes, about 30°, their diameter increases. Consequently the pressure gradient becomes less steep, and the tropical cyclone often changes into an extratropical cyclone.

The tropical cyclones do not show the characteristic distribution of temperature and precipitation associated with fronts which is found in the cyclones of temperate latitudes. Nevertheless, many writers believe now that tropical cyclones originate at frontal surfaces between air masses of different temperature and motion like the extratropical cyclones[2] and some observational evidence is cited to support this view.

[1] VON HANN, J., and SÜRING, R., "Lehrbuch der Meteorologie," 4th ed., p. 222, C. H. Tauchnitz, Leipzig, 1926.

[2] NORMAND, C. W. B., *Gerl. Beitr. Geophys.*, **34**, 233, 1931.

According to Rodewald[1] a place particularly favorable to the origin of tropical cyclones is a so-called "triple point," which is a point where three air masses meet. Deppermann[2] found in a study of 18 cyclones originating east of the Philippines that one-third were linked up with such a triple point.

It is very probable that any asymmetries of temperature which may have been present in the nascent state of a tropical cyclone escape observation; for the beginning of the storms that take place over the oceans is rarely observed, and a fairly dense network of stations is required to discover these asymmetries. If the tropical cyclones start as waves at a frontal surface, their development will be very similar to that of the extratropical cyclones described in Secs. 109 and 110, although probably much more rapid.

A peculiarity of many tropical cyclones is the calm center, the so-called "eye of the storm." When the center of a tropical cyclone passes over a given locality, the wind which has been very violent dies down suddenly, either to an absolute calm or at least to a much lower velocity, and the precipitation ceases. There have even been cases reported of the sky clearing. The diameter of the eye of the storm may be about 10 to 30 mi.

The frequency of the tropical cyclones is very small compared with the frequency of the extratropical cyclones. They occur almost exclusively during the warm season, whereas the cyclones of temperate latitudes are stronger and more frequent in winter.

We have already mentioned that some observations seem to indicate that tropical cyclones have warm sectors in their nascent stage and may thus start as waves at a frontal surface, analogously with the extratropical cyclones (Secs. 109 and 110). It is, however, quite possible that in tropical cyclones which appear in regions of strong vertical convection the cyclonic rotation is developed in the manner suggested by the convection theory (Sec. 112). The velocity distribution represented by (112.2) seems to be realized to a good degree of approximation in a considerable number of tropical cyclones.

The convergence causing the tropical cyclone is presumably started by the ascent of air in a region which in the subsequent

[1] Rodewald, M., *Met. Z.*, **53**, 516, 1936.

[2] Deppermann, C. E., Philippine Weather Bureau, Manila Central Observatory, 1939.

development of the storm becomes the central calm. Such an ascent of air must frequently occur in the tropics when the surface-air temperature has been raised considerably. Especially over water the air will be very moist. After having risen a short distance or even at the beginning of the ascent, it will cool not at the dry- but at the moist-adiabatic rate. Thus, the temperature difference between the central core and the outlying region may be maintained up to considerable heights. In general, however, showers are all that such ascent will lead to. Only if the ascent takes place over a sufficiently large region may the horizontally inflowing air travel sufficiently long distances for an appreciable cyclonic rotation to originate, owing to the conservation of angular momentum. It may also happen that with each local convection current a small rotation is set up and that a tropical cyclone forms when the conditions are favorable for the amalgamation of these small vortices. The conditions for the formation of a tropical cyclone instead of local showers appear to be fulfilled only rarely, for the observations show that tropical cyclones are rather infrequent phenomena.

With the aid of either the wave theory or the convection theory, it is easily explained why cyclones do not form close to the equator. The effect of the earth's rotation vanishes here. It is this effect that produces unstable waves by turning the plane of motion into the horizontal and that generates the rotational velocity. It is hardly necessary to state that not all tropical cyclones must originate in the same manner and that the convection theory may account for the formation of some tropical cyclones and the wave theory for others.

The air that converges toward the central region must flow out of the cyclone at higher levels, for the low surface pressure indicates a mass deficit. Durst and Sutcliffe[1] have pointed out that the horizontal pressure gradient must decrease with the altitude, for the cyclone does not reach throughout the whole atmosphere. The air ascending near the central core will therefore ascend to levels where its centrifugal force is too strong for the existing pressure gradient; consequently, the air obtains an outward component, and thus the low surface pressure near the center is created.

[1] Durst, C. S., and Sutcliffe, C. R., *Quart. J. Roy. Met. Soc.*, **64**, 75, 1938.

A satisfactory explanation of the calm center has not yet been given. Brunt[1] suggests that the wind energy near the center may be annihilated by the viscous dissipation of energy. Haurwitz[2] has shown that owing to the strong outward action of the centrifugal force the gradient-wind level is reached at considerably lower altitudes than in extratropical cyclones (see page 209). This effect is particularly strong close to the center of the vortex. It is evident that no ascending motion takes place in the eye of the storm, for no precipitation is observed here.

The height of tropical cyclones, that is, the level up to which an appreciable pressure gradient exists cannot be less than that of the extratropical cyclones. Otherwise, it would be necessary to assume very much higher temperatures near the central core of the storm than outside.[3] The rather scanty aerological data do not show such temperature increase.

116. Anticyclones. In some respects, anticyclones are the counterpart of cyclones; the pressure increases toward the center, and the circulation of the air is, roughly speaking, opposite to that found in cyclones. But even a superficial study shows that anticyclones are not simply cyclones with the opposite direction of pressure gradient and wind, as their name implies. On page 154, it was shown that the wind velocity and the pressure gradient in an anticyclone cannot exceed a certain limiting value which decreases toward the center if a balance is to exist between pressure-gradient force, Coriolis force, and centrifugal force. Any weather map shows that, actually, pressure gradients and winds, especially in the central part of anticyclones, are weak.

The air temperature at the surface is often lower in anticyclones than in cyclones. Before upper-air data were available the opinion was widely held that the low-pressure areas are due to the smaller weight of a column of warmer air, whereas the high-pressure areas are due to the greater weight of air at a lower than normal temperature. Hann, however, showed with the aid of observations from mountain observatories that this view is untenable and that as a rule anticyclones are warmer than cyclones. The following table, which contains values of pres-

[1] BRUNT, *op. cit.*, p. 305.

[2] HAURWITZ, B., *Gerl. Beitr. Geophys.*, **47**, 206, 1936.

[3] HAURWITZ, B., *Monthly Weather Rev.*, **63**, 45, 1935.

sure, temperature, and density for cyclones and anticyclones over Europe, published by W. H. Dines,[1] shows that, throughout the whole troposphere, cyclones are actually colder than anticyclones; only in the stratosphere is the opposite true. The temperature and density at the surface are not given, for in such a statistical investigation stations of different elevations are used. Moreover, the surface temperature is very much affected by the properties of the ground and is not representative of the true air temperature. The air temperature in the surface layers of anticyclones often sinks considerably, especially at night and during the winter, owing to the clear sky which allows a very strong radiational cooling. But these low temperatures are only surface phenomena in a very large group of anticyclones, and at a height of only 1 km the anticyclone is already warmer than the cyclone.

Pressure, Temperature, and Density in Cyclones and Anticyclones
(After W. H. Dines)

Height, km	Cyclone			Anticyclone		
	Pressure, mb	Temperature, deg abs	Density, gm/m³	Pressure, mb	Temperature, deg abs	Density, gm/m³
0	989	...	....	1026		
1	875	276	1105	908	279	1135
2	772	270	997	802	276	1010
3	678	263	898	708	271	906
4	594	256	808	623	265	818
5	519	249	724	547	259	734
6	451	242	648	478	253	658
7	390	234	582	417	246	589
8	337	228	516	362	238	530
9	289	226	446	313	231	472
10	249	225	389	269	225	417
11	214	224	333	231	220	366
12	184	225	285	197	217	316
13	158	225	245	168	215	272
14	135	224	210	143	215	232

[1] Dines, W. H., *Geophys. Mem.*, Met. Off., London, **13**, 69, 1919.

Hanzlik[1] found from a study of anticyclones over Europe that there are two types, cold and warm. The warm anticyclones occur more frequently over Europe; but it seems that, at least in winter, cold anticyclones are more frequent over North America. In warm air the pressure decreases less rapidly than in cold air. Consequently, warm anticyclones extend much higher than cold anticyclones. The latter are, in fact, mostly very shallow structures which extend only through the lower part of the troposphere. At greater heights a low pressure is frequently found over the surface high, as shown in the case of an anticyclone in polar air over North America by Haurwitz and Noble.[2] In this instance, a strong surface high over the western part of Canada and the northern United States had already disappeared at the 5000-ft level, and at 14,000 ft a strong center of low pressure was situated in about the same position as the surface high.

Khanewsky[3] and Runge,[4] who studied the origin of high anticyclones, found that there are often two air currents side by side, a colder north-easterly and a warmer south-westerly current. Runge suggested that, in the transitional zone between these two opposing currents, anticyclonic rotation may develop. The Coriolis force deflects each current to its right so that between the two currents the air must ascend vertically. The pressure will then increase until the pressure gradient balances the Coriolis force. Though Runge's explanation of the origin of the anticyclone is rather vague, it deserves attention because it does not derive the field of motion from the pressure field, as is all too frequently done, but attempts rather a dynamic explanation of the field of pressure as a consequence of the field of motion.

The origin of anticyclonic rotation as a banking effect on the right side of a current as suggested by Rossby has been discussed on page 236.

Another point that may be of importance in respect to the development of high anticyclones has been brought out by Durst.[5]

[1] Hanzlik, S., *Denkschrift. Wien. Akad. Wiss.*, **84**, 163, 1908; **88**, 67, 1912.

[2] Haurwitz, B., and Noble, J. R. H., *Bull. Am. Met. Soc.*, **19**, 107, 1938.

[3] Khanewsky, W., *Met. Z.*, **46**, 81, 1929.

[4] Runge, H., *Met. Z.*, **49**, 129, 1932.

[5] Durst, C. S., *Quart. J. Roy. Met. Soc.*, **59**, 231, 1933.

He found that the rotational velocity around the center of high pressure increases with altitude. Because subsidence occurs in the anticyclone, air with a higher velocity is brought down to levels where the pressure gradient and the comparatively small centrifugal force cannot balance the Coriolis force which is directed toward the center so that a component in this direction results. This inflow tends to compensate for the outflow due to friction near the ground. Consequently, the warm anticyclone may exist for a considerable time. But Brunt[1] points out that these warm anticyclones are not necessarily always very stable systems, and he quotes in particular a case of a well-established anticyclone of Sept. 15, 1932, which disappeared unexpectedly within 2 days.

The cold anticyclones are obviously due to the excess weight of the cold air. It should, however, be noted, also, that in the high, warm anticyclones the air is denser than in the high, cold cyclones, as can be seen from the table on page 338. The density depends not only on the temperature, but also on the pressure. The higher pressure in anticyclones more than compensates the effect of the higher temperature.

In the warm anticyclones the temperature in the stratosphere is below the average. It has been suggested that the greater weight of the cold stratospheric air produces the higher surface pressure in spite of the higher temperature in the lower layers. Southerly currents in the stratosphere would transport colder air northward and would cause a rise of the pressure at the surface, as described on pages 328 to 330 in connection with Stüve's hypothesis of cyclogenesis.

On the other hand, Bjerknes's theory of the coupling between the variations in the upper and lower layers of the atmosphere (page 330) explains how a low, cold anticyclone over a dome of cold air spreads higher up into the stratosphere, owing to the development of anticyclonic vorticity over the west side of the dome of cold air.

[1] Brunt, *op. cit.*, p. 381.

APPENDIX

TABLE I.—SATURATION PRESSURE OF WATER VAPOR, MILLIBARS
(After H. H. Landolt–R. Börnstein, "Physikalisch-chemische Tabellen," 5th ed., p. 1314, Verlag Julius Springer, Berlin, 1923)

a. Saturation Pressure over Water

Deg C	0	1	2	3	4	5	6	7	8	9
−10	2.86	2.64	2.44	2.25	2.07	1.91	1.75			
− 0	6.11	5.68	5.27	4.89	4.54	4.21	3.90	3.62	3.35	3 10
+ 0	6.11	6.57	7.06	7.58	8.13	8.72	9.35	10.02	10.73	11.48
10	12.28	13.13	14.02	14.98	15.98	17.05	18.18	19.38	20.64	21.97
20	23 38	24.87	26.44	28.09	29.84	31.68	33.61	35.65	37.80	40.06
30	42.45	44.93	47.55	50.31	53.20	56.23	59.42	62.76	67.26	69.92
40	74.10	77.79	82.00	86.40	91.01	95.84	100.87	106.1	111.6	117.4

b. Saturation Pressure over Ice

Deg C	0	1	2	3	4	5	6	7	8	9
−30	0.39	0.34	0.30	0.27	0.25	0.22	0.20	0.18	0.16	0.14
−20	1 03	0.94	0.85	0.77	0.70	0.63	0.57	0.51	0 46	0.42
−10	2.60	2.37	2.17	1.98	1.81	1.65	1.50	1.37	1.25	1.11
− 0	6.11	5.62	5.17	4.76	4.37	4.01	3.68	3.38	3.10	2.83

TABLE II.—NUMERICAL CONSTANTS

Equatorial radius of the earth....	6378.4 km
Polar radius of the earth.........	6356.9 km
Radius of the sphere having (approximately) the same area and volume as the earth...........	$E = 6371.2$ km
Angular velocity of the earth's rotation.....................	$\omega = 7.292 \times 10^{-5}$ sec^{-1}
Acceleration of gravity at sea level and 45° latitude..............	$g_0 = 980.621$ cm/sec^2
Universal gas constant...........	$R^* = 83.13 \times 10^6$ ergs/deg $= 1.986$ cal/deg
Molecular weight of the air......	$m = 28.97$
Gas constant for air.............	$R = 2.87 \times 10^6$ cm^2/sec^2 deg
Molecular weight of water vapor..	$m_w = 18$
Heat of condensation of water vapor.......................	$L = (595 - 0.5t°C)$ cal/gm
Specific heat of dry air at constant volume......................	$c_v = 0.170$ cal/gm deg
Specific heat of dry air at constant pressure.....................	$c_p = 0.239$ cal/gm deg
Ratio of the specific heats of air...	$\lambda = c_p/c_v = 1.405$
Mechanical equivalent of heat....	$J = 4.185 \times 10^7$ ergs/cal
Heat equivalent of work.........	$A = 1/J = 2.39 \times 10^{-8}$ cal/erg
Specific heat of water vapor at constant pressure................	$c_{pw} = 0.466$ cal/gm deg
Heat of fusion of water..........	$L_i = 79.7$ cal/gm
Specific heat of ice.............	$c_i = 0.49$ cal/gm deg
Heat of sublimation.............	$L_s = 677$ cal/gm
Wien's constant (law of displacement).......................	$a = 0.2892$ cm deg
Stefan-Boltzmann constant.......	$\sigma = 5.70 \times 10^{-5}$ erg cm^{-2} sec^{-1} deg^{-4} $= 0.817 \times 10^{-10}$ cal/cm^{-2} min^{-1} deg^{-4}
Solar constant..................	$I_0 = 1.94$ cal/cm^2 min
Coefficient of molecular viscosity of the air at 0°C and 760 mm Hg	$\mu = 1.71 \times 10^{-4}$ gm cm^{-1} sec^{-1}

NAME INDEX

SUBJECT INDEX

SOLUTIONS TO PROBLEMS

1. According to (6.22),

$$\frac{e}{e_0} = \left(\frac{T}{T_0}\right)^{\frac{g}{R\alpha}\frac{m_w}{m}}$$

Thus, the following values are obtained for z, T, and e:

z, km	0	1	2	3	4	5	6	7	8
T, deg abs	288	282	276	270	264	258	252	246	240
e, mb	10	9.3	8.6	8.0	7.4	6.8	6.2	5.7	5.2

Already at 2 km the computed vapor pressure would be larger than the saturation pressure according to Table I! (Concerning the general impossibility of the establishment of Dalton's law in a nonisothermal atmosphere even in the absence of mixing see also R. Emden, "Thermodynamik der Himmelskörper," p. 402, B. G. Teubner, Leipzig and Berlin, 1926, reprinted from Encyklopädie der mathematischen Wissenschaften, Vol. 6, (2), p. 24.)

2. According to (6.22) and (4.1),

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{g}{R\alpha}-1}$$

The autoconvective lapse rate $= g/R = 3.41°C/100$ m. Such lapse rates are observed only next to the surface of the earth during times of strong insolation. It should be noted that the atmosphere is still in unstable *equilibrium*, even if the lapse rate is autoconvective and that a disturbance will always lead to a rearrangement of the air layers when the original lapse rate was greater than the adiabatic, not greater than the autoconvective lapse rate.

3. If p_0 is the surface pressure, T_0 the surface temperature, and ρ_0 the surface density,

$$p_0 = R\rho_0 T_0$$

Because ρ_0 is the density of the homogeneous atmosphere at every level,

$$p_0 = g\rho_0 H$$

where H is the height of the homogeneous atmosphere. Regarding g as constant,

$$H = \frac{RT_0}{g}$$

The same result is obtained from Prob. 2 by computing the height at which the temperature has fallen to absolute zero if the lapse rate of the temperature is autoconvective. If $T_0 = 283°$ abs, $H = 8280$ m.

4. According to (6.21) and (4.1),

$$\rho = \frac{p_0}{RT} e^{-\frac{gz}{RT}}$$

Differentiating logarithmically with respect to the time,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{T}\frac{\partial T}{\partial t} + \frac{gz}{RT^2}\frac{\partial T}{\partial t}$$

Thus $$\frac{\partial \rho}{\partial t} = 0 \qquad \text{where} \qquad z = \frac{RT}{g}$$

5. From (6.21) and (6.22)

$$\ln p_0 = \ln p_H + \frac{g}{R}\frac{H - h}{T_0 - \alpha h} + \frac{g}{R\alpha} \ln T_0 - \frac{g}{R\alpha} \ln (T_0 - \alpha h)$$

By differentiation and rearrangement,

$$\frac{dp_0}{p_0} = \frac{g}{R}\frac{\alpha(H - h)}{(T_0 - \alpha h)^2} dh$$

A variation of the height of the tropopause as assumed in the problem may be connected with a considerable variation of the surface pressure (see B. Haurwitz, in *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **3**, 272, 1927).

6. Assume a constant mean temperature T_m up to the level h to which the outbreak of cold air extends. Assume further that the pressure above this level remains constant. By logarithmic differentiation of (6.21),

$$\frac{dp_0}{p_0} = -\frac{g}{R}\frac{h}{T_m^2} dT_m$$

If the variation of the mean temperature dT_m is not known from aerological data, it will often be possible to estimate it from observations of the surface temperature. The height h of the outbreak of cold air (or of a mass of warm air) can then be computed from the above formula (see also Sec. 65).

7. According to (23.1), $\dfrac{d^2z}{dt^2} = g\dfrac{T' - T}{T}$

where T' is the temperature of the moving parcel, and T of the surrounding air. From (9.2),

$$\frac{dT'}{dz} = -\Gamma \frac{T'}{T}$$

(Note that in Sec. 9 the meaning of T' and T was reversed). By integration of the last equation,

$$\frac{T'}{T_0'} = \left(\frac{T}{T_0}\right)^{\frac{\Gamma}{\alpha}}$$

where T_0' and T_0 are the temperature of the parcel and the air surrounding it at the starting level, $z = 0$. Upon substituting the last relation in (23.1), it follows that

$$\frac{d^2z}{dt^2} = g\left[\frac{T_0'}{T_0}\left(\frac{T}{T_0}\right)^{\frac{\Gamma}{\alpha}-1} - 1\right] = g\,\frac{T_0' - T_0}{T_0} - g\,\frac{T_0'}{T_0}\,\frac{\Gamma - \alpha}{T_0}\,z$$

if the exponential is expanded according to the binomial theorem and only first-order terms are retained. Consider first $\Gamma > \alpha$.
Putting

$$A^2 = g\,\frac{T_0'}{T_0}\,\frac{\Gamma - \alpha}{T_0}$$
$$z = \frac{T_0' - T_0}{\Gamma - \alpha}\,\frac{T_0}{T_0'}\,(1 - \cos At)$$

where the constants of integration are chosen so that $z = 0$ and $dz/dt = 0$, when $t = 0$. The amplitude is larger, the larger the initial temperature difference between the moving parcel of air and the air surrounding it, and larger, the smaller the difference between the existing and the adiabatic lapse rate. The length of the period $2\pi/A$ is inversely proportional to $\sqrt{\Gamma - \alpha}$. If $\Gamma < \alpha$, putting

$$A^{*2} = g\,\frac{T_0'}{T_0}\,\frac{\alpha - \Gamma}{T_0}$$
$$z = \frac{T_0' - T_0}{\alpha - \Gamma}\,\frac{T_0}{T_0'}\,(\cosh A^*t - 1)$$

If $\Gamma = \alpha$,

$$z = g\,\frac{T_0' - T_0}{T_0}\,t^2$$

When $\Gamma \leqq \alpha$, z increases or decreases continuously with time so that the motion is unstable, whereas it is stable when $\Gamma > \alpha$.

The practical application of these calculations is severely restricted, for only a simple parcel of air is considered, without any reference to the motion of the surrounding air. A more adequate treatment is possible by hydrodynamical methods.

8. Subtract the second of Eqs. (40.1) from the first, integrate over the whole spectrum, and note the condition of radiation equilibrium (40.2).

9. According to (44.1),

$$r^2(\dot\lambda + \omega)\cos^2\varphi = \text{const}$$

Upon differentiating logarithmically, it follows that, with sufficient accuracy,

$$d\dot\lambda = -2(\dot\lambda + \omega)\,\frac{dr}{r}$$

If $dr > 0$,

$$d\dot\lambda < 0, \text{ motion toward the west}$$

If $dr < 0$,

$$d\lambda > 0, \text{ motion toward the east}$$

The resulting velocities are comparatively small.

10. If α is the constant vertical lapse rate of temperature, it follows from (6.22) that

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{g}{R\alpha}}$$

This is the geometric equation from which it is seen that the atmosphere is barotropic. In order to have autobarotropy, the physical equation must be a piezotropic equation identical with the preceding geometric equation. If a polytropic equation of the form (8.5), $p/p_0 = (T/T_0)^{\frac{1}{\bar{\kappa}}}$, is chosen as physical equation, the atmosphere is autobarotropic provided that

$$\bar{\kappa} = \frac{R\alpha}{g}$$

and

$$c = c_p - \frac{g}{R\alpha}(c_p - c_v)$$

11. Let $V(r)$ be the velocity of the circular flow. Consider the circulation along the following closed curve in the fluid: from a point A along a circle of radius r_1 to a point B (length of the arc AB χr_1 where χ is the angle that the radii through A and B make at the center), then along the radius through B to a point C at a distance $r_1 - r_2$ from B ($r_1 > r_2$), and from C along the circle of radius r_2 to the point D on the radius through A (length of the arc χr_2) and back to A. The circulation along this path is $\chi r_1 V(r_1) - \chi r_2 V(r_2)$, for the velocity is perpendicular to the radius. Because the motion is irrotational when the circulation vanishes, the statement made in the problem is correct.

12*a*. With the aid of the gas equation (4.1), it follows from (52.21) that

$$\oint \frac{dp}{\rho} = R \iint \frac{1}{p}\left(\frac{\partial T}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial p}{\partial x}\right) dx\, dy$$

Substituting from (9.1),

$$\oint \frac{dp}{\rho} = \frac{R}{p^{\kappa}} \iint p^{\kappa-1}\left(\frac{\partial \Theta}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \Theta}{\partial y}\frac{\partial p}{\partial x}\right) dx\, dy$$

The last expression can easily be changed into a form analogous to the integral in (52.3).

12*b*. Upon eliminating the pressure in the last equation by (9.1), it follows that

$$\oint \frac{dp}{\rho} = \frac{R}{\kappa} \iint \frac{1}{\Theta}\left(\frac{\partial \Theta}{\partial x}\frac{\partial T}{\partial y} - \frac{\partial \Theta}{\partial y}\frac{\partial T}{\partial x}\right) dx\, dy$$

These two forms for the circulation integral may often be more suitable for a discussion of atmospheric circulations than the expression containing the gradients of pressure and specific volume.

13. The components of the thermal wind at a level z are under the assumption suggested in the problem, according to (55.31) and (55.32),

$$u - u_0 = -\frac{gz}{2\omega \sin \varphi} \frac{1}{T} \frac{\partial T}{\partial y}$$
$$v - v_0 = \frac{gz}{2\omega \sin \varphi} \frac{1}{T} \frac{\partial T}{\partial x}$$

Thus, the velocity of the thermal wind

$$v_{th} = \frac{gz}{2\omega \sin \varphi} \frac{1}{T} \frac{\Delta T}{\Delta n}$$

where ΔT is the temperature difference between two consecutive isotherms and Δn their distance. According to (53.21) the geostrophic wind

$$v_g = \frac{1}{\rho 2\omega \sin \varphi} \frac{\Delta p}{\Delta n}$$

and

$$\frac{v_{th}}{v_g} = \frac{gz}{RT^2} p \frac{\Delta T}{\Delta p}$$

Thus, in order to find the thermal wind, the wind velocity may be read off the geostrophic scale, the isotherms being treated as if they were isobars. The figure obtained for the wind velocity has to be multiplied by the factor given by the preceding equation. The thermal wind is parallel to the isotherms and in such a direction that the colder air is to the left and the warmer air to the right if one faces in the direction of the wind (in the Northern Hemisphere). To find the geostrophic wind at the level z, the thermal wind has to be added vectorially to the geostrophic wind at the surface. (For a detailed discussion, see E. M. Vernon and E. V. Ashburn, *Monthly Weather Rev.*, **66**, 267, 1938.)

14. The barotropic relation may be given in the form $\rho = \rho(p)$. Then, upon differentiating (53.11) with respect to z, it follows that

$$2\omega \sin \varphi \frac{\partial v}{\partial z} = -\frac{1}{\rho^2} \frac{d\rho}{dp} \frac{\partial p}{\partial z} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{\partial p}{\partial x} \right)$$

With the aid of (53.13),

$$2\omega \sin \varphi \frac{\partial v}{\partial z} = +\frac{1}{\rho} \frac{d\rho}{\partial p} g \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial x} (-g\rho)$$
$$= \frac{1}{\rho} \frac{d\rho}{dp} g \frac{\partial p}{\partial x} - \frac{1}{\rho} g \frac{d\rho}{dp} \frac{\partial p}{\partial x} = 0$$

In the same manner, it is found that $\partial u/\partial z = 0$ in a barotropic atmosphere.

15. According to (53.21),

$$2\omega \sin \varphi\, v = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Along the isentropic surface (denoted by the suffix Θ),

$$\left(\frac{\partial p}{\partial x}\right)_\Theta = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial \Phi}\left(\frac{d\Phi}{dx}\right)_\Theta = \frac{\partial p}{\partial x} - \rho\left(\frac{d\Phi}{dx}\right)_\Theta$$

Φ is the geopotential, and $(d\Phi/dx)_\Theta$ the variation of the geopotential of the isentropic surface in the x-direction. According to (9.1), along an isentropic surface,

$$\kappa \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_\Theta = \frac{1}{T}\left(\frac{\partial T}{\partial x}\right)_\Theta$$

Thus,

$$2\omega \sin \varphi v = \left\{\frac{d\,[(c_p/A)\,T + \Phi]}{dx}\right\}_\Theta$$

(See R. B. Montgomery, *Bull. Am. Met. Soc.*, **18**, 210, 1937, and A. F. Spilhaus, *ibid.*, **21**, 239, 1940, for details.)

16. According to (56.5) and (53.23),

$$\frac{dz}{dr} = \frac{1}{g}\left(\frac{v^2}{r} + 2\omega \sin \varphi\, v\right)$$

where the index of v is omitted. Because the velocity is continuous at R,

$$v = \eta r \qquad \text{when} \qquad r < R$$
$$v = \eta \frac{R^2}{r} \qquad \text{when} \qquad r > R$$

If z_0 is the height of the isobaric surface at the center, $r = 0$

$$z = \frac{\eta^2 + 2\omega \sin \varphi\, \eta}{2g} r^2 + z_0$$

in the inner part of the vortex, $r < R$. When an isobaric surface ends at the earth's surface, the integration constant z_0 is negative. In the outer part of the vortex,

$$z = \frac{\eta^2}{2g} R^2 \left(1 - \frac{R^2}{r^2}\right) + \frac{2\omega \sin \varphi}{g} \eta\, R^2 \ln \frac{r}{R} + z_1$$

The integration constant z_1 stands for the height of the isobaric surface at the distance R from the center. Because the isobaric surface must be continuous across the circle of radius R,

$$z_1 = z_0 + \frac{\eta^2 + 2\omega \sin \varphi}{2g} \eta\, R^2$$

17. Let Δp be the pressure interval between two consecutive isobars, $\Delta \dot{p}$ the tendency interval between two consecutive isallobars, and Δn the distance between the isobars and the isallobars, respectively. Then, according to (53.21), the geostrophic wind

$$v_g = \frac{1}{2\omega \sin \varphi} \frac{1}{\rho} \frac{\Delta p}{\Delta n}$$

According to (57.6) the isallobaric wind

$$v' = \frac{1}{(2\omega \sin \varphi)^2} \frac{1}{\rho} \frac{\Delta \dot{p}}{\Delta n}$$

and

$$\frac{v'}{v_g} = \frac{1}{2\omega \sin \varphi} \frac{\Delta \dot{p}}{\Delta p}$$

Thus, in order to find the isallobaric wind, the wind velocity may be read off the geostrophic scale, the isallobars being treated as if they were isobars. The value obtained for the wind velocity has to be multiplied by the conversion factor $\frac{1}{2\omega \sin \varphi} \frac{\Delta \dot{p}}{\Delta p}$. The direction of the isallobaric wind is toward the greatest fall of pressure (see E. Gold, *Quart. J. Roy. Met. Soc.*, **61**, 127, 1935).

18. From (58.2), it follows that

$$\frac{\partial p_0}{\partial t} = -p_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Upon choosing the x-axis in the direction of the wind, $\partial p_0/\partial t = -p_0\, \partial u/\partial x$. Because $\partial p_0/\partial t = 1$ mb/3 hr and $p_0 = 1000$ mb, it is found that

$$\frac{\partial u}{\partial x} = 10^{-7} \text{ sec}^{-1} = 1 \text{ cm/sec/100 km}.$$

Such a small variation of the wind cannot be observed.

19. According to the equation of continuity (47.3),

$$-\rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = + \frac{\partial \rho w}{\partial z}$$

Integrating between the surface and the level h,

$$-\frac{p_0 - p_h}{g} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \rho_h w_h$$

Choosing the x-axis in the direction of the wind,

$$w_h = -\frac{p_0 - p_h}{g\rho_h} \frac{\partial u}{\partial x}$$

Let $p_0 - p_h = 100$ mb, $\rho_h = 10^{-3}$ gm/cm^3. Because $w_h = 10$ cm/sec or larger,

$$\frac{\partial u}{\partial x} = 10^{-4} \text{ sec}^{-1} = 10 \text{ m/sec/100 km, or larger}$$

20. According to (60.33),

$$\frac{dy}{dx} = -\frac{(\partial p/\partial x)_1 - (\partial p/\partial x)_2}{(\partial p/\partial y)_1 - (\partial p/\partial y)_2}$$

Note that x is the direction toward the east. Upon substituting from the geostrophic wind relation (53.11) and (53.12), it follows that

$$\tan \psi = \frac{dy}{dx} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 u_1 - \rho_2 u_2} = \frac{T_2 v_1 - T_1 v_1}{T_2 u_1 - T_1 u_2}$$

ψ denotes the angle between the front and the E-direction, and v and u are the N- and E-components of the wind.

21. If (56.5) and (60.31) are used, it follows that

$$\frac{dz}{dr} = \frac{2\omega \sin \varphi(v_1\rho_1 - v_2\rho_2) + (\rho_1 v_1^2 - \rho_2 v_2^2)/r}{g(\rho_1 - \rho_2)}$$

Here the index θ of the tangential velocity component has been omitted and the small term due to the vertical component of the Coriolis force in the denominator has been neglected.

22. Let $-ku$, $-kv$ be the components of the frictional force, k being a proportionality factor. Then, instead of the geostrophic equations (53.21) and (53.22), the equilibrium conditions become

$$-2\omega \sin \varphi\, v + ku = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$2\omega \sin \varphi\, u + kv = 0$$

Note that the x-direction coincides with the (positive) pressure gradient. It follows that

$$V = \sqrt{v^2 + u^2} = \frac{1}{\sqrt{(2\omega \sin \varphi)^2 + k^2}}\frac{1}{\rho}\frac{\partial p}{\partial x}$$

The wind velocity is now smaller than without the effect of friction. The angle between wind velocity and the x-axis

$$\tan (V,x) = \frac{v}{u} = -\frac{2\omega \sin \varphi}{k}$$

Thus, now the wind does not blow parallel to the isobars but has a component toward lower pressure, $u = -\dfrac{k}{(2\omega \sin \varphi)^2 + k^2}\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ (see C. M. Guldberg and H. Mohn, "Études sur les mouvements de l'atmosphère," I, A. W. Brögger, Oslo, 1876).

23. According to (76.7), after some trigonometric transformations,

$$\sqrt{u_0^2 + v_0^2} = v_g(\cos \alpha_0 - \sin \alpha_0)$$

24. If F_x and F_y denote the components of the frictional force, it follows from (76.4) that

$$F_x = \frac{\mu}{\rho}\frac{d^2u}{dz^2} = -2\omega \sin \varphi\, v'$$
$$F_y = \frac{\mu}{\rho}\frac{d^2v}{dz^2} = 2\omega \sin \varphi\, u$$

Note that near the surface the frictional force must be in the fourth quadrant, for $F_x > 0$, and $F_y < 0$. Equation (76.7) shows that

$$\tan (F, x) = \frac{F_y}{F_x} = -\frac{u}{v'} = -\tan\left(az + \alpha_0 - \frac{\pi}{4}\right).$$

Thus,

$$(F, x) = -\left(az + \alpha_0 - \frac{\pi}{4}\right)$$

for the other possible solution $(F, x) = \pi - \left(az + \alpha_0 - \frac{\pi}{4}\right)$ would imply that the frictional force near the earth's surface is in the second quadrant.

Because α_0 is the angle between the negative pressure gradient (negative x-axis) and the wind, the angle between the frictional force and the direction opposite to the wind at the surface is $\pi/4$ (see D. Brunt, *Quart. J. Roy. Met. Soc.*, **46**, 174, 1920).

At the gradient-wind level, $(F, x) = -\pi$. The frictional force at this level is in the direction of the negative x-axis, whereas the wind is in the direction of the positive y-axis.

Thus the simple assumption of Guldberg and Mohn on which Prob. 22 is based is not fulfilled.

25. Because $\partial^2 v_g/\partial z^2$ vanishes, there is no frictional wind component under these assumptions above the gradient-wind level, except for the insignificant component given by Eq. (76.7).

26. In the steady case the amount of dust s sinking with the velocity c through the unit area per unit time must be equal to the amount of dust transported upward by turbulent mass exchange,

$$-\rho cs - A\frac{ds}{dz} = 0$$

a. When A = const,

$$s = s_0 e^{-\frac{c\rho}{A}z}$$

b. When $A = k\,(z + a)$,

$$\frac{s}{s_0} = \left(\frac{z+a}{a}\right)^{-\frac{c\rho}{k}}$$

27. According to (84.4),

$$\Theta = \Theta_0 + \gamma z - (\Theta_0 - \Theta_1)\left\{1 - E\left[\frac{z}{\sqrt{4(A/\rho)t}}\right]\right\}$$

where Θ_0 and Θ_1 are the potential temperatures of the air at the ground before and after cooling begins. If T is the temperature, T_0 and T_1 the surface temperatures before and after the cooling begins, α the lapse rate of T, and $\alpha = \Gamma - \gamma$, approximately,

$$T = T_0 - \alpha z - (T_0 - T_1)\left[1 - E\left(\frac{z}{\sqrt{4(A/\rho)t}}\right)\right] \qquad \text{approximately.}$$

The temperature is greatest at the height where

$$\frac{dT}{dz} = 0$$

It follows that

$$z^2_{\max} = 4\frac{A}{\rho}t \ln\left[\frac{T_0 - T_1}{\alpha}\frac{1}{\sqrt{\pi(A/\rho)t}}\right]$$

It is easily seen that this condition gives the height of maximum temperature, not of minimum temperature. A maximum temperature exists as long as

$$t \leqq \frac{(T_0 - T_1)^2}{\alpha^2}\frac{1}{\pi(A/\rho)}$$

Later the temperature decreases everywhere from the ground upward.

Upon putting $dz^2_{\max}/dt = 0$, it is found that $z_{\max}$ is highest at the time

$$\left(\frac{T_0 - T_1}{\alpha}\right)^2 \frac{\rho}{\pi A e}$$

At this instant,

$$z_{\max} = \sqrt{\frac{2}{\pi e}}\,\frac{T_0 - T_1}{\alpha}$$

and

$$T = T_0 - (T_0 - T_1)\left[\sqrt{\frac{2}{\pi e}} + 1 - E\left(\sqrt{\frac{1}{2}}\right)\right] = T_0 - 0.86(T_0 - T_1)$$

28. Let $\dfrac{A}{\rho} = a_0(1 + \epsilon \cos \nu t)$. If $\tau = t + \dfrac{\epsilon}{\nu}\sin \nu t$,

the equation for turbulent mass exchange becomes

$$\frac{\partial T}{\partial \tau} = a_0 \frac{\partial^2 T}{\partial z^2}$$

and it follows according to (83.3) that

$$T = Be^{-\lambda z}\cos(\nu t + \epsilon \sin \nu t - \lambda z) \qquad \text{where} \qquad \lambda = \sqrt{\frac{\nu}{2a_0}}$$

29. According to (88.2) and (6.22),

$$I^* = \frac{Jc_v}{g + R\alpha}(T_0 p_0 - T_h p_h)$$

To obtain an approximation, substitute again from (6.22),

$$I^* = \frac{Jc_v}{g + R\alpha} T_0 p_0 \left[1 - \left(1 - \frac{\alpha h}{T_0}\right)^{\frac{g}{R\alpha}+1}\right]$$

Upon developing into a power series and omitting terms of higher than the second degree in $\alpha h/T_0$, it follows that

$$I^* = \frac{1}{\lambda - 1} p_0 h \left(1 - \frac{1}{2}\frac{g}{R}\frac{h}{T_0}\right)$$

30. The solution for each layer is of the form (101.6). Let the internal surface of discontinuity be where $z = 0$ and let h be the depth of the lower fluid. Then $w = 0$, when $z = -h$. Thus,

$$C_1 e^{2\pi\frac{h}{\lambda}} + C_2 e^{-2\pi\frac{h}{\lambda}} = 0$$

Upon introducing a new constant K, it follows that, in the lower layer,

$$\begin{aligned} w &= K \sinh 2\pi \frac{z+h}{\lambda} \sin \frac{2\pi}{\lambda}(x - ct) \\ p &= \rho c K \cosh 2\pi \frac{z+h}{\lambda} \cos \frac{2\pi}{\lambda}(x - ct) \end{aligned}$$

and in the upper layer, because $w = 0$ where $z = h'$,

$$\begin{aligned} w' &= K' \sinh 2\pi \frac{z-h'}{\lambda} \sin \frac{2\pi}{\lambda}(x - ct) \\ p' &= \rho' c K \cosh 2\pi \frac{z-h'}{\lambda} \cos \frac{2\pi}{\lambda}(x - ct) \end{aligned}$$

Upon applying the boundary condition (100.8) (note that we may put in the perturbation quantities $z = 0$ as a sufficiently good approximation), it follows that

$$\frac{2\pi}{\lambda} c^2 \left(\rho K \cosh 2\pi \frac{h}{\lambda} - \rho' K' \cosh 2\pi \frac{h'}{\lambda}\right) - g(\rho - \rho') K \sinh 2\pi \frac{h}{\lambda} = 0$$

and

$$\frac{2\pi}{\lambda} c^2 \left(\rho K \cosh 2\pi \frac{h}{\lambda} - \rho' K' \cosh 2\pi \frac{h'}{\lambda}\right) + g(\rho - \rho') K' \sinh 2\pi \frac{h'}{\lambda} = 0$$

Because the ratio K/K' must be the same in both equations and, for small values of h/λ, $\coth 2\pi \frac{h}{\lambda} = \frac{1}{2\pi}\frac{\lambda}{h}$,

$$c^2 \frac{\rho'}{h'} + c^2 \frac{\rho}{h} - g(\rho - \rho') = 0$$

and

$$c = \sqrt{g \frac{\rho - \rho'}{\rho h' + \rho' h} h'h}$$

31. In the undisturbed state, $g = -\frac{1}{\rho}\frac{\partial P}{\partial z}$. The perturbation equations are, according to (99.5) and (99.6),

$$\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0
\end{aligned}$$

Let $U = U_0 + bz$ where U_0 is the undisturbed velocity at the rigid lower boundary, $z = 0$. Assume that

$$\begin{aligned}
u &= A(z)\cos\frac{2\pi}{\lambda}(x - ct) \\
w &= C(z)\sin\frac{2\pi}{\lambda}(x - ct) \\
p &= D(z)\cos\frac{2\pi}{\lambda}(x - ct)
\end{aligned}$$

Upon substituting the expressions for u, w, and p in the perturbation equations, if follows that

$$A = \frac{\lambda}{2\pi}C', \qquad D = \frac{\lambda}{2\pi}\rho[(c - U)C' + bC]$$

and

$$(c - U)C'' - \left(\frac{2\pi}{\lambda}\right)^2 (c - U)C = 0$$

Excluding the possibility that $c - U = 0$,

$$C = K_1 e^{2\pi\frac{z}{\lambda}} + K_2 e^{-2\pi\frac{z}{\lambda}}$$

or, because $w = 0$ at the rigid lower boundary where $z = 0$,

$$C = K \sinh 2\pi\frac{z}{\lambda}$$

The height of the fluid layer may be denoted by h. Then the equation of the free upper surface is given by

$$P + p = -g\rho(z - h) + K\frac{\lambda}{2\pi}\rho\left[b\sinh 2\pi\frac{h}{\lambda} + (c - U_h)\frac{2\pi}{\lambda}\cosh 2\pi\frac{h}{\lambda}\right] \cos\frac{2\pi}{\lambda}(x - ct) = 0$$

Here h has been substituted for z in the term for p (see page 280). Upon applying the boundary condition (100.8) it follows that

$$(c - U_h)^2 + (c - U_h)\frac{\lambda b}{2\pi}\tanh 2\pi\frac{h}{\lambda} - \frac{g\lambda}{2\pi}\tanh 2\pi\frac{h}{\lambda} = 0$$

where h has been substituted for z in w. Thus,

$$c = U_h - \frac{\lambda b}{4\pi}\tanh 2\pi\frac{h}{\lambda} \pm \sqrt{\frac{\lambda^2 b^2}{4^2\pi^2}\left(\tanh 2\pi\frac{h}{\lambda}\right)^2 + \frac{g\lambda}{2\pi}\tanh 2\pi\frac{h}{\lambda}}$$

If the layer is very shallow,

$$c = U_h - \frac{bh}{2} \pm \sqrt{\left(\frac{bh}{2}\right)^2 + gh}$$

If the layer is very deep,

$$c = U_h - \frac{\lambda b}{4\pi} \pm \sqrt{\frac{b^2\lambda^2}{4^2\pi^2} + \frac{g\lambda}{2\pi}}$$

On introducing plausible values for b in these expressions, it will be found that the effect of the vertical variation of the undisturbed current is in general very slight. (See B. Haurwitz, *Veröffentlich. Geophys. Inst. Leipzig*, 2d ser., **5**, 67, 1931.)